Parametricity for Haskell with Imprecise Error Semantics

Florian Stenger[1] and Janis Voigtländer

Technische Universität Dresden

TLCA'09

[1]Supported by the DFG under grant VO 1512/1-1.

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Applications

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Reasoning in Haskell: An Example

$$\begin{array}{llll}
\texttt{takeWhile} :: (\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha] & & & \\
\texttt{takeWhile}\ p\ [\,] & & = [\,] & \\
\texttt{takeWhile}\ p\ (a : as) & \mid\ p\ a & = a : (\texttt{takeWhile}\ p\ as) & \\
 & \mid\ \texttt{otherwise} & = [\,] &
\end{array}$$

Reading Haskell: An Example

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For every choice of p, f, and l:

$$\texttt{takeWhile}\ p\ (\texttt{map}\ f\ l) \quad = \quad \texttt{map}\ f\ (\texttt{takeWhile}\ (p \circ f)\ l)$$

Provable by induction.

Reasoning in Haskell: An Example

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\begin{array}{l}
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Or as a "free theorem" [Wadler, FPCA'89].

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Reasoning in Haskell: An Example

$$\texttt{takeWhile} :: (\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha]$$

$$\texttt{filter} :: (\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha]$$

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\texttt{g}\ p\ (\texttt{map}\ f\ l) &= \texttt{map}\ f\ (\texttt{g}\ (p \circ f)\ l)
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Errors in Haskell

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Traditionally, all error causes subsumed under "$\perp$".

Errors in Haskell

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  in `loop`

Traditionally, all error causes subsumed under "$\perp$".

Better, explicit distinction. Like:

$$\begin{array}{rcl} \mathit{Ok}\ v & : & \text{nonerroneous} \\ \mathit{Bad}\ \text{“}\cdots\text{”} & : & \text{finitely failing} \\ \perp & : & \text{nonterminating} \end{array}$$

Naive Propagation of Errors

- `tail` $[1/0, 2.5] \rightsquigarrow \mathit{Ok}\ ((\mathit{Ok}\ 2.5) : (\mathit{Ok}\ []))$

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- $(\texttt{error}\ s)\ (\cdots) \ \rightsquigarrow\ \mathit{Bad}\ s$
- $\textbf{case}\ (\texttt{error}\ s)\ \textbf{of}\ \{\cdots\} \ \rightsquigarrow\ \mathit{Bad}\ s$
- $(\texttt{error}\ s_1) + (\texttt{error}\ s_2) \ \rightsquigarrow\ ???$

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- $\textbf{case}\ (\texttt{error}\ s)\ \textbf{of}\ \{\cdots\} \ \leadsto \ Bad\ s$
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Dependence on evaluation order leads to considerably less freedom for implementors to rearrange computations, to optimise!

Imprecise Error Semantics [Peyton Jones et al., PLDI'99]

Basic idea:

$$
\begin{aligned}
\mathit{Ok}\ v &: \text{nonerroneous} \\
\mathit{Bad}\ \{\cdots\} &: \text{finitely failing, nondeterministic} \\
\bot &: \text{nonterminating}
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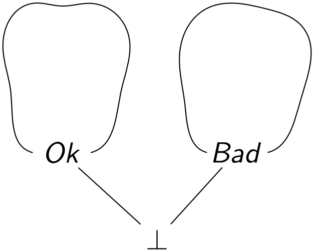
Definedness order:



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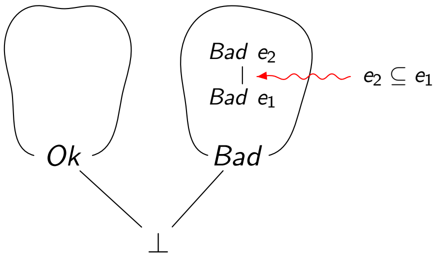

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Actual Propagation of Errors:

- $(\texttt{error}\ s_1) + (\texttt{error}\ s_2) \ \rightsquigarrow\ \mathit{Bad}\ \{s_1, s_2\}$

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Actual Propagation of Errors:

- $(\texttt{error}\ s_1) + (\texttt{error}\ s_2) \ \rightsquigarrow\ \mathit{Bad}\ \{s_1, s_2\}$

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- $\textbf{case}\ (\texttt{error}\ s_1)\ \textbf{of}\ \{(x, y) \rightarrow \texttt{error}\ s_2\} \ \rightsquigarrow\ \mathit{Bad}\ \{s_1, s_2\}$

Impact on Program Equivalence

"Normally":

$$\texttt{takeWhile}\ p\ (\texttt{map}\ f\ l) \quad = \quad \texttt{map}\ f\ (\texttt{takeWhile}\ (p \circ f)\ l)$$

where:

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But now:

$$\texttt{takeWhile}\ \texttt{null}\ (\texttt{map}\ \texttt{tail}\ (\texttt{error}\ s))$$

$$\neq$$

$$\texttt{map}\ \texttt{tail}\ (\texttt{takeWhile}\ (\texttt{null} \circ \texttt{tail})\ (\texttt{error}\ s))$$

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Because:

$$\texttt{takeWhile}\ (\texttt{null} \circ \texttt{tail})\ (\texttt{error}\ s) \ \rightsquigarrow\ \mathit{Bad}\ \{s, \text{“empty list”}\}$$

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Because:

$$\texttt{takeWhile}\ (\texttt{null} \circ \texttt{tail})\ (\texttt{error}\ s)\ \leadsto\ \mathit{Bad}\ \{s, \text{“empty list”}\}$$

while:

$$\texttt{takeWhile}\ \texttt{null}\ (\texttt{map}\ \texttt{tail}\ (\texttt{error}\ s))\ \leadsto\ \mathit{Bad}\ \{s\}$$

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Because:

$$\texttt{takeWhile}\ (\texttt{null} \circ \texttt{tail})\ (\texttt{error}\ s) \ \rightsquigarrow\ \mathit{Bad}\ \{s, \text{“empty list”}\}$$

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Now, imagine this in the following program context:

$$\begin{array}{l}\texttt{catchJust}\ \texttt{errorCalls}\ (\texttt{evaluate}\ \cdots)\\ \qquad (\lambda s \rightarrow \textbf{if}\ s == \text{“empty list”}\\ \qquad\qquad \textbf{then}\ \texttt{return}\ [[42]]\\ \qquad\qquad \textbf{else}\ \ \texttt{return}\ [\,])\end{array}$$

How to Revise Free Theorems?

[Wadler, FPCA'89] : for every $\texttt{g} :: (\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha]$,

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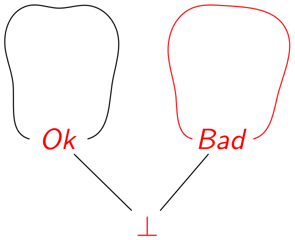
What are corresponding conditions "in real"?



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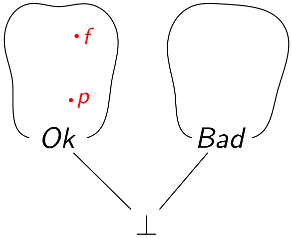

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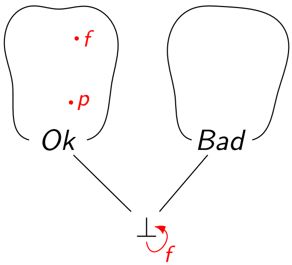

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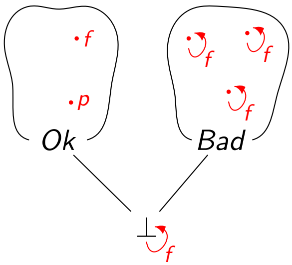

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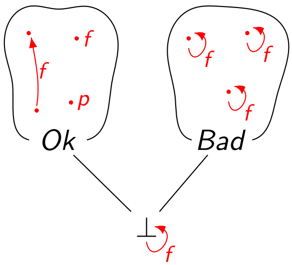

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