Type-Based Reasoning for Real Languages

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University of Bonn

PPL'10

Parametric Polymorphism in Haskell

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\texttt{takeWhile} :: (\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha] \\
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For every choice of p, f, and l:

$$\texttt{takeWhile}\ p\ (\texttt{map}\ f\ \ l) \quad = \quad \texttt{map}\ f\ (\texttt{takeWhile}\ (p \circ f)\ l)$$

Provable by induction.

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- That is what was claimed!

Automatic Generation of Free Theorems

At `http://www-ps.iai.uni-bonn.de/ft`:

This tool allows to generate free theorems for sublanguages of Haskell as described here.

The source code of the underlying library and a shell-based application using it is available here and here.

Please enter a (polymorphic) type, e.g. "(a -> Bool) -> [a] -> [a]" or simply "filter":

`g :: (a -> Bool) -> [a] -> [a]`

Please choose a sublanguage of Haskell:

- (•) no bottoms (hence no general recursion and no selective strictness)
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- (•) equational
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[Generate]

Automatic Generation of Free Theorems

The theorem generated for functions of the type

```
g :: forall a . (a -> Bool) -> [a] -> [a]
```

in the sublanguage of Haskell with no bottoms is:

```
forall t1,t2 in TYPES, R in REL(t1,t2).
 forall p :: t1 -> Bool.
  forall q :: t2 -> Bool.
   (forall (x, y) in R. p x = q y)
   ==> (forall (z, v) in lift{[]}(R).
         (g p z, g q v) in lift{[]}(R))
```

The structural lifting occurring therein is defined as follows:

```
lift{[]}(R)
  = {([], [])}
  u {(x : xs, y : ys) |
      ((x, y) in R) && ((xs, ys) in lift{[]}(R))}
```

Reducing all permissible relation variables to functions yields:

```
forall t1,t2 in TYPES, f :: t1 -> t2.
 forall p :: t1 -> Bool.
  forall q :: t2 -> Bool.
   (forall x :: t1. p x = q (f x))
   ==> (forall y :: [t1]. map f (g p y) = g q (map f y))
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Export as PDF   Show type instantiations   Enter a new type   Help page

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But this also allows a $\mathtt{g}$ with

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To prevent this, we have to compare

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- Call $(x_1, x_2) :: (\mathsf{Bool}, \mathsf{Bool})$ and $(y_1, y_2) :: (\mathsf{Int}, \mathsf{Int})$ related if $(x_1, y_1) \in \mathcal{R}$ and $(x_2, y_2) \in \mathcal{R}$.

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- Then g_{Bool} and g_{Int} with

$$\begin{aligned} g_{\mathsf{Bool}}\ (x, y) &= \texttt{not}\ x \\ g_{\mathsf{Int}}\ (x, y) &= y + 1 \end{aligned}$$

  are not related for choice of, e.g., $\mathcal{R} = \{(\mathsf{True}, 1)\}$.

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- Call $(x_1, x_2) :: (\text{Bool}, \text{Bool})$ and $(y_1, y_2) :: (\text{Int}, \text{Int})$ related if $(x_1, y_1) \in \mathcal{R}$ and $(x_2, y_2) \in \mathcal{R}$.
- Call $f_1 :: (\text{Bool}, \text{Bool}) \to \text{Bool}$, $f_2 :: (\text{Int}, \text{Int}) \to \text{Int}$ related if related inputs lead to related outputs.
- Then g_{Bool} and g_{Int} with

$$\begin{aligned} g_{\text{Bool}}\ (x, y) &= \mathtt{not}\ x \\ g_{\text{Int}}\ (x, y) &= y + 1 \end{aligned}$$

  are not related for choice of, e.g., $\mathcal{R} = \{(\text{True}, 1)\}$.

Reynolds: $g :: \tau$, with α free in τ, iff for every $\tau_1, \tau_2, \mathcal{R} \subseteq \tau_1 \times \tau_2$, g_{τ_1} is related to g_{τ_2} by the "propagation" of $\mathcal{R}$ (replaced for α) along τ.

Deriving Free Theorems, in General

For interpreting types as relations:

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$$(\mathcal{R}, \mathcal{S}) \quad = \quad \{((x_1, x_2), (y_1, y_2)) \mid (x_1, y_1) \in \mathcal{R}, (x_2, y_2) \in \mathcal{S}\}$$

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Then for every $\mathrm{g} :: \tau$, the pair (g, g) is contained in the relational interpretation of τ.

Now Formal Counterpart to Intuitive Reasoning

Let $\mathrm{g} :: (\alpha \to \mathsf{Bool}) \to ([\alpha] \to [\alpha])$.

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by definition of $\forall \mathcal{R}. \mathcal{F}(\mathcal{R})$

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\Rightarrow\ & \forall (a_1, a_2) \in (f \to id_{\mathsf{Bool}}).\ \forall (l_1, l_2) \in (\mathtt{map}\ f). \\
& \qquad\qquad\qquad\qquad (\mathrm{g}_{\tau_1}\ a_1\ l_1, \mathrm{g}_{\tau_2}\ a_2\ l_2) \in (\mathtt{map}\ f)
\end{aligned}
$$

by instantiating $\mathcal{R} = f$ and realising that then $[\mathcal{R}] = (\mathtt{map}\ f)$

for every function $f :: \tau_1 \to \tau_2$

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\Rightarrow\ & \forall (l_1, l_2) \in (\mathtt{map}\ f).\ (\mathtt{g}_{\tau_1}\ (p \circ f)\ l_1, \mathtt{g}_{\tau_2}\ p\ l_2) \in (\mathtt{map}\ f) \\
& \text{by instantiating } (a_1, a_2) = (p \circ f, p) \in (f \to id_{\mathsf{Bool}})
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for every function $f :: \tau_1 \to \tau_2$ and predicate $p :: \tau_2 \to \mathsf{Bool}$.

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by inlining

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That is what was claimed!

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The Polymorphic λ-Calculus [Girard 1972, Reynolds 1974]

Types: $\tau := \alpha \mid \tau \to \tau \mid \forall\alpha.\tau$

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(\mathcal{R},\mathcal{S}) & = & \{((x_1,x_2),(y_1,y_2)) \mid (x_1,y_1)\in\mathcal{R},(x_2,y_2)\in\mathcal{S}\} \\
[\mathcal{R}] & = & \{([x_1,\ldots,x_n],[y_1,\ldots,y_n]) \mid n\geq 0,(x_i,y_i)\in\mathcal{R}\} \\
\mathcal{R}\rightarrow\mathcal{S} & = & \{(f_1,f_2) \mid \forall(a_1,a_2)\in\mathcal{R}.\ (f_1\ a_1,f_2\ a_2)\in\mathcal{S}\} \\
\forall\mathcal{R}.\mathcal{F}(\mathcal{R}) & = & \{(u,v) \mid \forall\tau_1,\tau_2,\mathcal{R}\subseteq\tau_1\times\tau_2.\ (u_{\tau_1},v_{\tau_2})\in\mathcal{F}(\mathcal{R})\}
\end{array}
$$

Then for every $\mathrm{g} :: \tau$, the pair (g,g) is contained in the relational interpretation of τ.

General Recursion

We had that for every

$$\texttt{g} :: (\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha]$$

it holds

$$\texttt{g}\ p\ (\texttt{map}\ f\ l) \quad = \quad \texttt{map}\ f\ (\texttt{g}\ (p \circ f)\ l)$$

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$$\begin{aligned} &\texttt{g} :: (\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha] \\ &\texttt{g}\ p\ l = [\texttt{head}\ (\texttt{g}\ p\ l)] \end{aligned} \qquad ?$$

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The above free theorem fails!

Consider, e.g., $p = \texttt{id}$, $f = \texttt{const}$ True, and $l = [\,]$.

Why $\texttt{g}\ p\ (\texttt{map}\ f\ l) = \texttt{map}\ f\ (\texttt{g}\ (p \circ f)\ l)$, Intuitively

- $\texttt{g} :: (\alpha \to \text{Bool}) \to [\alpha] \to [\alpha]$ must work uniformly for every instantiation of α.
- The output list can only contain elements from the input list l.
- Which, and in which order/multiplicity, can only be decided based on l and the input predicate p.
- The only means for this decision are to inspect the length of l and to check the outcome of p on its elements.
- The lists $(\texttt{map}\ f\ l)$ and l always have equal length.
- Applying p to an element of $(\texttt{map}\ f\ l)$ always has the same outcome as applying $(p \circ f)$ to the corresponding element of l.
- $\texttt{g}$ with p always chooses "the same" elements from $(\texttt{map}\ f\ l)$ for output as does $\texttt{g}$ with $(p \circ f)$ from l, except that in the former case it outputs their images under f.
- $\texttt{g}\ p\ (\texttt{map}\ f\ l)$ is equivalent to $\texttt{map}\ f\ (\texttt{g}\ (p \circ f)\ l)$.
- That is what was claimed!

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- $\texttt{g}\ p\ (\texttt{map}\ f\ l)$ is equivalent to $\texttt{map}\ f\ (\texttt{g}\ (p \circ f)\ l)$, if f is strict.
- This gives a revised free theorem.

The Polymorphic λ-Calculus [Girard 1972, Reynolds 1974]

Types: $\tau := \alpha \mid \tau \to \tau \mid \forall\alpha.\tau \mid \mathsf{Bool} \mid [\tau]$

Terms: $t := x \mid \lambda x : \tau.t \mid t\ t \mid \Lambda\alpha.t \mid t\ \tau \mid$
$\mathsf{True} \mid \mathsf{False} \mid [\,]_\tau \mid t : t \mid \textbf{case}\ t\ \textbf{of}\ \{\cdots\}$

$$\Gamma, x : \tau \vdash x : \tau \qquad \frac{\Gamma, x : \tau_1 \vdash t : \tau_2}{\Gamma \vdash (\lambda x : \tau_1.t) : \tau_1 \to \tau_2}$$

$$\frac{\Gamma \vdash t : \tau_1 \to \tau_2 \qquad \Gamma \vdash u : \tau_1}{\Gamma \vdash (t\ u) : \tau_2} \qquad \frac{\alpha, \Gamma \vdash t : \tau}{\Gamma \vdash (\Lambda\alpha.t) : \forall\alpha.\tau}$$

$$\frac{\Gamma \vdash t : \forall\alpha.\tau}{\Gamma \vdash (t\ \tau') : \tau[\tau'/\alpha]} \qquad \frac{\Gamma \vdash t : \tau \qquad \Gamma \vdash u : [\tau]}{\Gamma \vdash (t : u) : [\tau]}$$

$$\Gamma \vdash \mathsf{True} : \mathsf{Bool} \quad , \quad \Gamma \vdash \mathsf{False} : \mathsf{Bool} \quad , \quad \Gamma \vdash [\,]_\tau : [\tau]$$

$$\frac{\Gamma \vdash t : \mathsf{Bool} \qquad \Gamma \vdash u : \tau \qquad \Gamma \vdash v : \tau}{\Gamma \vdash (\textbf{case}\ t\ \textbf{of}\ \{\mathsf{True} \to u\,;\mathsf{False} \to v\}) : \tau}$$

$$\frac{\Gamma \vdash t : [\tau'] \qquad \Gamma \vdash u : \tau \qquad \Gamma, x_1 : \tau', x_2 : [\tau'] \vdash v : \tau}{\Gamma \vdash (\textbf{case}\ t\ \textbf{of}\ \{[] \to u\,;(x_1 : x_2) \to v\}) : \tau}$$

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To provide semantics, types are interpreted as pointed complete partial orders, and:

$$\textbf{fix}\ t \quad = \quad \bigsqcup_{i \geq 0} (t^i\ \bot)$$

Use in an Example

The function

$$\begin{array}{l}
\texttt{filter} :: (\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha] \\
\texttt{filter}\ p\ [\,] \qquad\quad = [\,] \\
\texttt{filter}\ p\ (a : as) = \textbf{if}\ p\ a\ \textbf{then}\ a : (\texttt{filter}\ p\ as) \\
\qquad\qquad\qquad\qquad\quad\ \ \textbf{else}\ \ \texttt{filter}\ p\ as
\end{array}$$

has a “desugaring” in the (extended) calculus as follows:

$$\begin{array}{l}
\textbf{fix}\ (\lambda f : (\forall\alpha.(\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha]). \\
\quad \Lambda\alpha.\lambda p : (\alpha \to \mathsf{Bool}).\lambda l : [\alpha]. \\
\quad \textbf{case}\ l\ \textbf{of}\ \{[\,] \qquad \to [\,]_\alpha\,; \\
\qquad\qquad\quad (a : as) \to \textbf{case}\ p\ a\ \textbf{of} \\
\qquad\qquad\qquad\qquad\quad \{\mathsf{True} \to a : (f\ \alpha\ p\ as)\,; \\
\qquad\qquad\qquad\qquad\quad\ \ \mathsf{False} \to f\ \alpha\ p\ as\}\})
\end{array}$$

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Let us check the one for, essentially, **fix** $:: (\alpha \to \alpha) \to \alpha$, namely:

$$\begin{array}{l} \forall \tau_1, \tau_2, \mathcal{R} \subseteq \tau_1 \times \tau_2. \forall t_1 :: \tau_1 \to \tau_1, t_2 :: \tau_2 \to \tau_2. \\ \qquad (\forall (a_1, a_2) \in \mathcal{R}.\ (t_1\ a_1, t_2\ a_2) \in \mathcal{R}) \\ \quad \Rightarrow (\textbf{fix}\ t_1, \textbf{fix}\ t_2) \in \mathcal{R} \end{array}$$

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We can guarantee the above, provided all relations are restricted to be strict and continuous.

Deriving Free Theorems in Presence of General Recursion

For interpreting types as relations:

1. Replace (implicit quantification over) type variables by (explicit) quantification over relation variables.
2. Replace types without any polymorphism by identity relations.
3. Use the following rules:

$$
\begin{array}{lcl}
(\mathcal{R},\mathcal{S}) & = & \{(\bot,\bot)\} \cup \{((x_1,x_2),(y_1,y_2)) \mid \cdots\} \\
[\mathcal{R}] & = & \{(\bot,\bot)\} \cup \{([x_1,\ldots,x_n],[y_1,\ldots,y_n]) \mid \cdots\} \\
\mathcal{R} \to \mathcal{S} & = & \{(f_1,f_2) \mid \forall (a_1,a_2) \in \mathcal{R}.\ (f_1\ a_1, f_2\ a_2) \in \mathcal{S}\} \\
\forall \mathcal{R}.\mathcal{F}(\mathcal{R}) & = & \{(u,v) \mid \forall \tau_1,\tau_2,\mathcal{R}\ \text{strict and continuous}.\ \cdots\}
\end{array}
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Then for every $\mathrm{g} :: \tau$, the pair (g, g) is contained in the (adapted) relational interpretation of τ.

Automatic Generation of Free Theorems

At `http://linux.tcs.inf.tu-dresden.de/~voigt/ft`:

This tool allows to generate free theorems for sublanguages of Haskell as described here.

The source code of the underlying library and a shell-based application using it is available here and here.

Please enter a (polymorphic) type, e.g. "(a -> Bool) -> [a] -> [a]" or simply "filter":

`g :: (a -> Bool) -> [a] -> [a]`

Please choose a sublanguage of Haskell:

- no bottoms (hence no general recursion and no selective strictness)
- general recursion but no selective strictness
- general recursion and selective strictness

Please choose a theorem style (without effect in the sublanguage with no bottoms):

- equational
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Generate

Adding Selective Strictness

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There are counterexamples, again.

Without **seq**, $\texttt{g}\ p\ (\texttt{map}\ f\ l) = \texttt{map}\ f\ (\texttt{g}\ (p \circ f)\ l)$

- $\texttt{g} :: (\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha]$ must work uniformly.
- The output list can only contain elements from the input list l and $\bot$.
- Which, and in which order/multiplicity, can only be decided based on l and the input predicate p.
- The only means for this decision are to inspect the length of l and to check the outcome of p on its elements and on $\bot$.
- The lists $(\texttt{map}\ f\ l)$ and l always have equal length.
- Applying p to an element of $(\texttt{map}\ f\ l)$ always has the same outcome as applying $(p \circ f)$ to the corresponding element of l.
- Applying p to $\bot$ has the same outcome as applying $(p \circ f)$, provided f is strict.
- $\texttt{g}$ with p always chooses "the same" elements from $(\texttt{map}\ f\ l)$ for output as does $\texttt{g}$ with $(p \circ f)$ from l, except that in the former case it outputs their images under f, and that they may also choose, at the same positions, to output $\bot$.
- $\texttt{g}\ p\ (\texttt{map}\ f\ l) = \texttt{map}\ f\ (\texttt{g}\ (p \circ f)\ l)$, if f is strict.

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Let us try the same strategy as before, looking at the free theorem for, essentially, $\textbf{seq} :: \alpha \to \beta \to \beta$, namely:

$$\forall \mathcal{R}, \mathcal{S} \text{ strict and continuous.}$$
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Case distinction:

t_1	t_1'	**seq** t_1 t_2	**seq** t_1' t_2'	$\in \mathcal{S}$
$\perp$	$\perp$	$\perp$	$\perp$	
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(Equational) Free Theorems in the Presence of **seq** [Johann & V., POPL'04]

For interpreting types as relations:

1. Replace (implicit quantification over) type variables by (explicit) quantification over relation variables.
2. Replace types without any polymorphism by identity relations.
3. Use the following rules:

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\begin{array}{lcl}
(\mathcal{R}, \mathcal{S}) & = & \{(\bot, \bot)\} \cup \{((x_1, x_2), (y_1, y_2)) \mid \cdots\} \\
[\mathcal{R}] & = & \{(\bot, \bot)\} \cup \{([x_1, \ldots, x_n], [y_1, \ldots, y_n]) \mid \cdots\} \\
\mathcal{R} \to \mathcal{S} & = & \{(f_1, f_2) \mid (f_1 = \bot \Leftrightarrow f_2 = \bot) \wedge \cdots\} \\
\forall \mathcal{R}. \mathcal{F}(\mathcal{R}) & = & \{(u, v) \mid \forall \tau_1, \tau_2, \mathcal{R} \text{ strict, continuous,} \\
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Then for every $g :: \tau$, the pair (g, g) is contained in the (adapted) relational interpretation of τ.

Revising Free Theorems

[Wadler, FPCA'89] : for every $\mathtt{g} :: (\alpha \to \mathrm{Bool}) \to [\alpha] \to [\alpha]$,

$$\mathtt{g}\ p\ (\mathtt{map}\ f\ l) \quad = \quad \mathtt{map}\ f\ (\mathtt{g}\ (p \circ f)\ l)$$

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[Christiansen et al., PLPV'10] : functional logic programs in Curry

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Systematic approach: replace

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Gain: Even if relations for un-Pointed types not strict anymore, free theorems continue to hold! [Launchbury & Paterson, ESOP'96]

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For example, we get:

- For every $\texttt{g} :: \mathsf{Pointed}\ \alpha \Rightarrow (\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha]$,

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- For every $\texttt{g} :: (\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha]$ (in the new system),

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  without conditions on f.

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- I ask: why must f be strict? What if it were not?
- The system gives me concrete $\texttt{g}$, as well as p, l, and (non-strict) f that refute the thus naivified free theorem.

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For example, search for a g such that

$$\text{Pointed } \alpha \vdash \mathrm{g} : (\alpha \to \text{Bool}) \to [\alpha] \to [\alpha]$$

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Problem: For term search, rules are not “syntax-directed” enough.

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$$\text{Pointed } \alpha \vdash \mathrm{g} : (\alpha \to \text{Bool}) \to [\alpha] \to [\alpha]$$

but not

$$\alpha \vdash \mathrm{g} : (\alpha \to \text{Bool}) \to [\alpha] \to [\alpha]$$

Natural first rule:

$$\frac{\Gamma \vdash \tau \notin \text{Pointed}}{\Gamma \Vdash (\textbf{fix } (\lambda x : \tau . x)) : \tau}$$

Otherwise, search further depending on type.

Problem: For term search, rules are not “syntax-directed” enough.

Particularly:

$$\frac{\Gamma \Vdash t : \tau_1 \to \tau_2 \qquad \Gamma \Vdash u : \tau_1}{\Gamma \Vdash (t \ u) : \tau_2}$$

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  [Seidel & V., FLOPS'10]

The Tool on an Example

The Free Theorem

The theorem generated for functions of the type

```
f :: (a -> Int) -> Int
```

is:

```
forall t1,t2 in TYPES, g :: t1 -> t2, g strict.
 forall p :: t1 -> Int.
  forall q :: t2 -> Int.
   (forall x :: t1. p x = q (g x)) ==> (f p = f q)
```

The Counterexample

By disregarding the strictness condition on `g` the theorem becomes wrong. The term

```
f = (\x1 -> (x1 _|_))
```

is a counterexample.

By setting `t1 = t2 = ... = ()` and

```
g = const ()
```

the following would be a consequence of the thus "naivified" free theorem:

```
(f p) = (f q)

where

p        = (\x1 -> 0)
q        = (\x1 -> (case x1 of {() -> 0}))
```

But this is wrong since with the above `f` it reduces to:

```
0 = _|_
```

Another Example

The Free Theorem

The theorem generated for functions of the type

```
f :: [a] -> Int
```

is:

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forall t1,t2 in TYPES, g :: t1 -> t2, g strict.
 forall x :: [t1]. f x = f (map g x)
```

The Counterexample

Disregarding the strictness condition on `g` the algorithm found no counterexample.

Question 1, for (**fix** and) **seq**

Are all totality and "$\neq \bot$"- conditions necessary for every g ?

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Problem: Completely new approach needed due to complications with function types.

(Equational) Free Theorems in the Presence of **seq** [Johann & V., POPL'04]

For interpreting types as relations:

1. Replace (implicit quantification over) type variables by (explicit) quantification over relation variables.
2. Replace types without any polymorphism by identity relations.
3. Use the following rules:

$$\begin{array}{lcl}
(\mathcal{R}, \mathcal{S}) & = & \{(\bot, \bot)\} \cup \{((x_1, x_2), (y_1, y_2)) \mid \cdots\} \\
[\mathcal{R}] & = & \{(\bot, \bot)\} \cup \{([x_1, \ldots, x_n], [y_1, \ldots, y_n]) \mid \cdots\} \\
\mathcal{R} \to \mathcal{S} & = & \{(f_1, f_2) \mid (f_1 = \bot \Leftrightarrow f_2 = \bot) \wedge \cdots\} \\
\forall \mathcal{R}. \mathcal{F}(\mathcal{R}) & = & \{(u, v) \mid \forall \tau_1, \tau_2, \mathcal{R} \text{ strict, continuous,} \\
& & \qquad \text{and bottom-reflecting. } \cdots\}
\end{array}$$

Then for every $g :: \tau$, the pair (g, g) is contained in the (adapted) relational interpretation of τ.

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Problem: Completely new approach needed due to complications with function types.

. . . But it Can be Done [Seidel & V., ATPS'09]

At `http://www-ps.iai.uni-bonn.de/cgi-bin/polyseq.cgi`:

The term

```
t = (/\a.
     (/\b.
      (\c::(a -> (b -> a)).
       (fix (\h::(a -> ([b] -> a)).
             (\n::a.
              (\ys::[b].
               (seq (c n) (case ys of {[] -> n; x:xs ->
                           (seq xs (seq x (let n' = ((c n) x) in
                                          ((h n') xs))))}))))))))))
```

can be typed to the optimal type

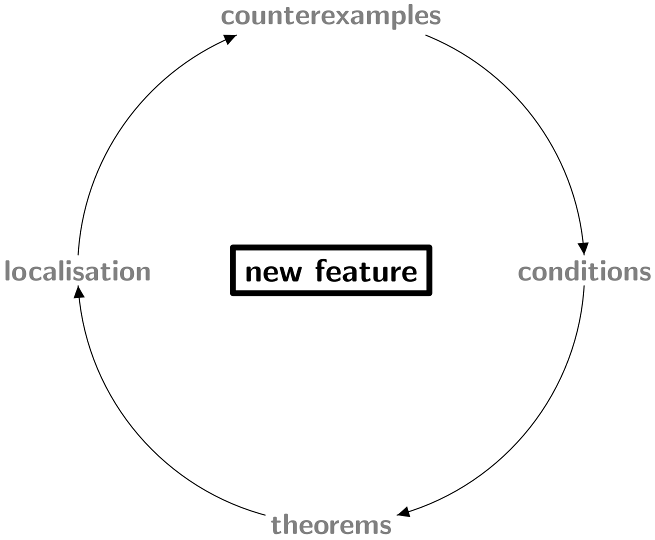
```
(forall^n a. (forall^e b. ((a ->^n (b ->^e a)) ->^e (a ->^e ([b] ->^e a)))))
```

with the free theorem

```
forall t1,t2 in TYPES, f :: t1 -> t2, f strict.
 forall t3,t4 in TYPES, g :: t3 -> t4, g strict and total.
  ((t_{t1}_{t3} /= _|_) <=> (t_{t2}_{t4} /= _|_))
  && (forall p :: t1 -> (t3 -> t1).
       forall q :: t2 -> (t4 -> t2).
        (forall x :: t1.
          ((p x /= _|_) <=> (q (f x) /= _|_))
          && (forall y :: t3. f (p x y) = q (f x) (g y)))
        ==> (((t_{t1}_{t3} p /= _|_) <=> (t_{t2}_{t4} q /= _|_))
             && (forall z :: t1.
                  ((t_{t1}_{t3} p z /= _|_) <=> (t_{t2}_{t4} q (f z) /= _|_))
                  && (forall v :: [t3].
                       f (t_{t1}_{t3} p z v) = t_{t2}_{t4} q (f z) (map_{t3}_{t4} g v)))))
```

The normal free theorem for the type without marks would be:

Investigating the Impact of a New Feature



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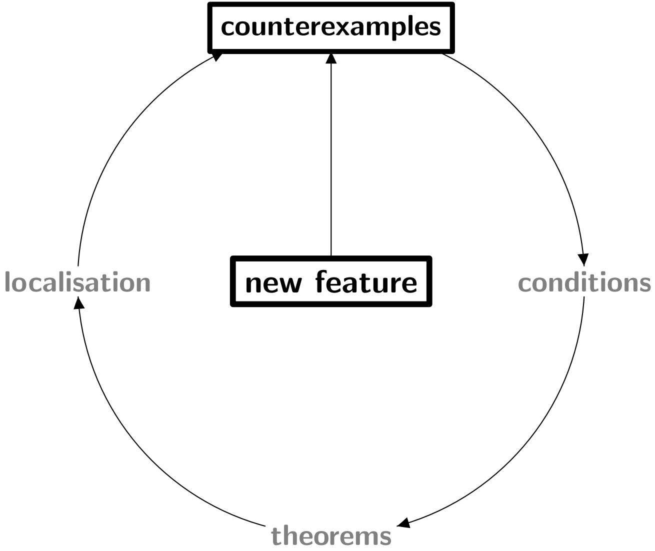

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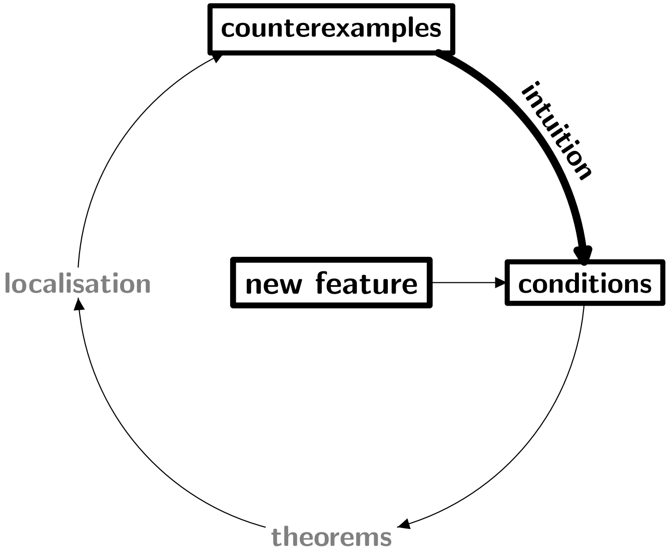

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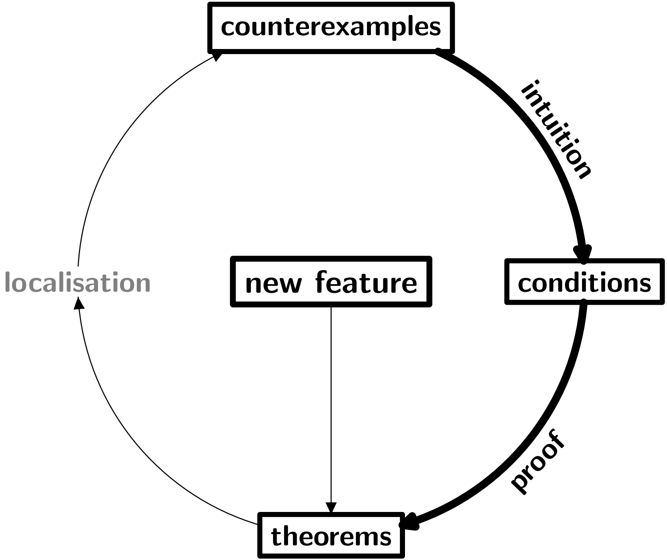

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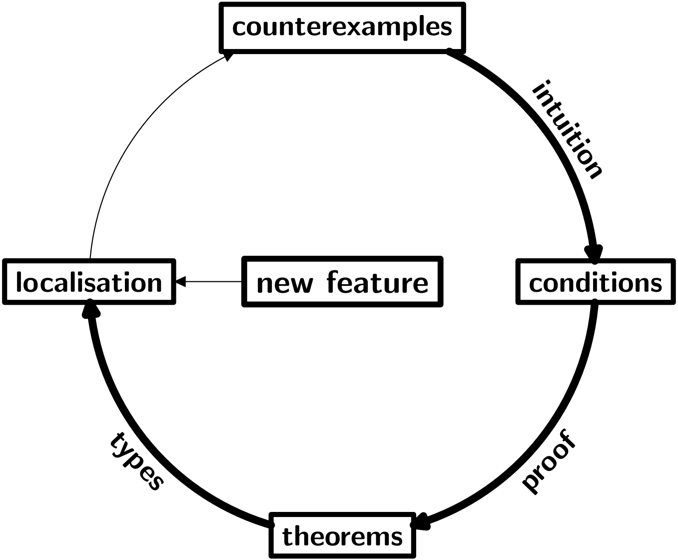

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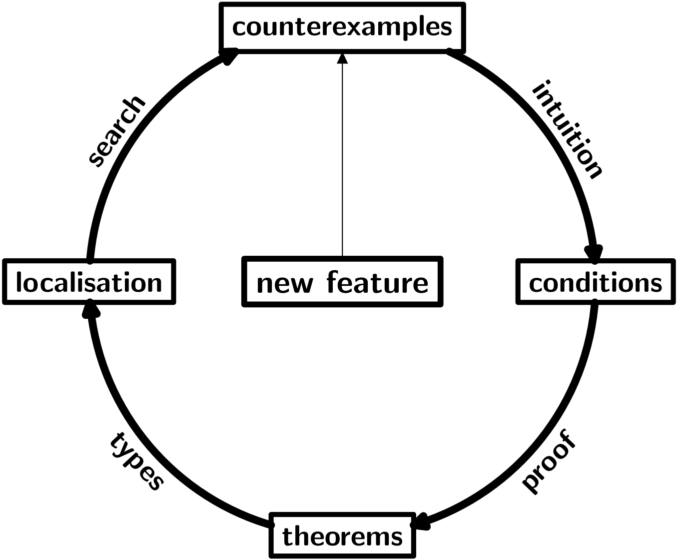
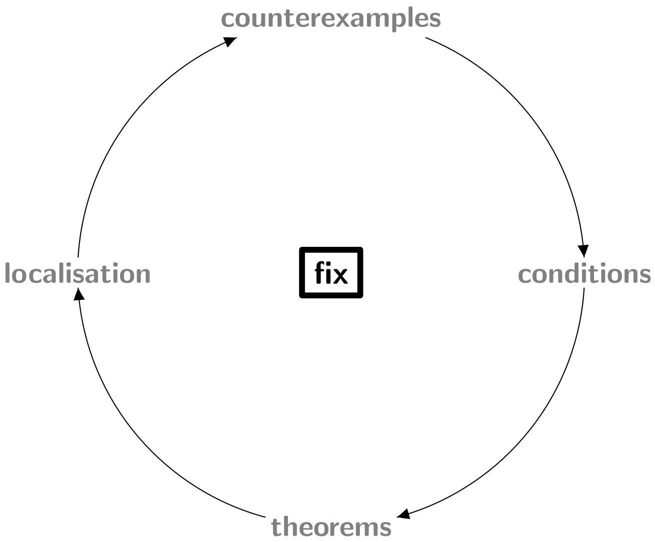


Progress for General Recursion



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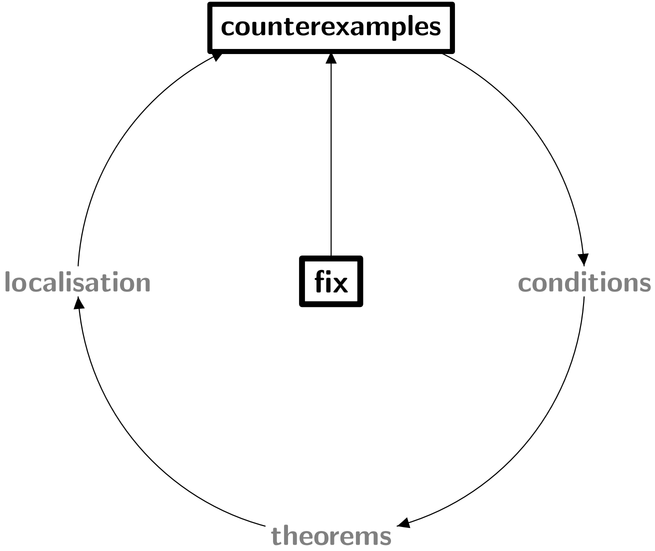

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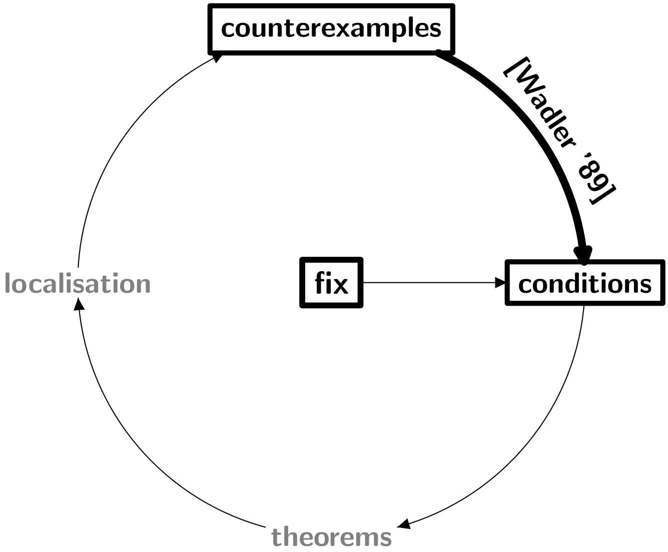

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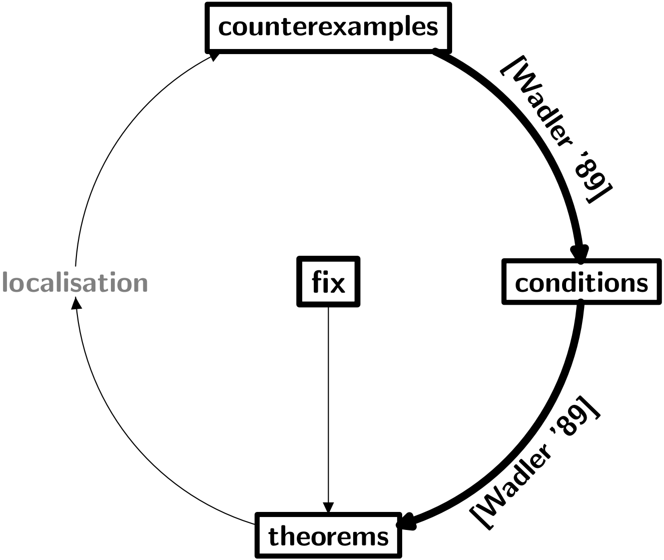

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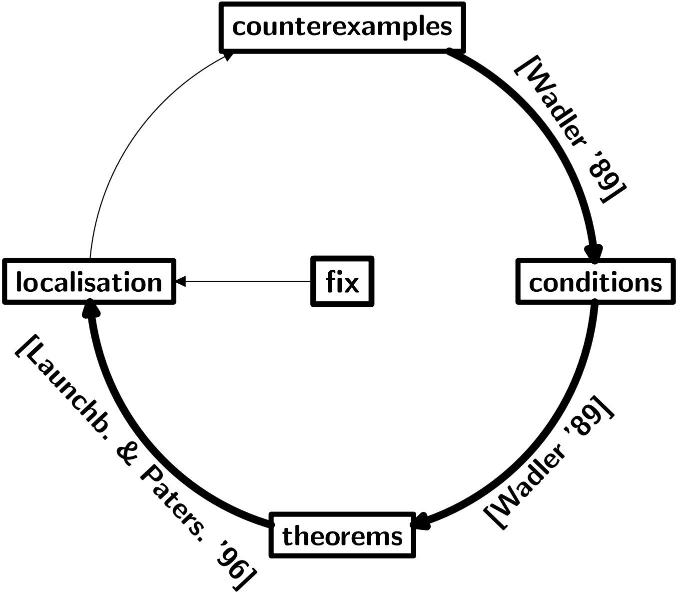

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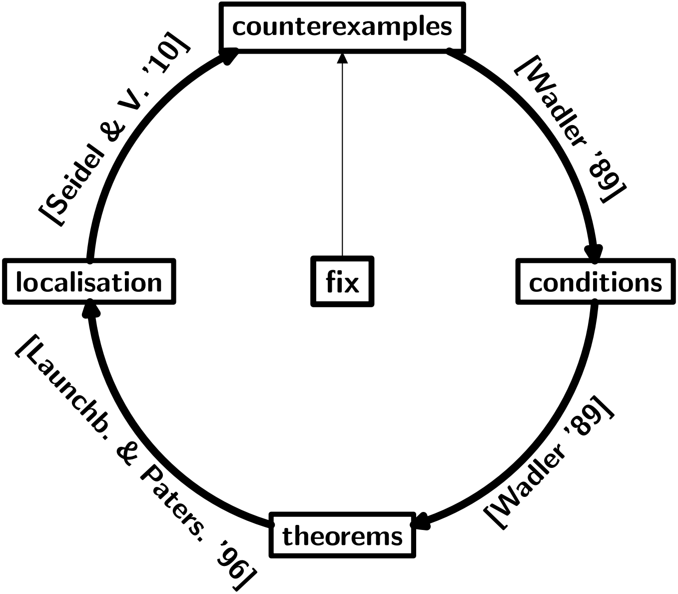
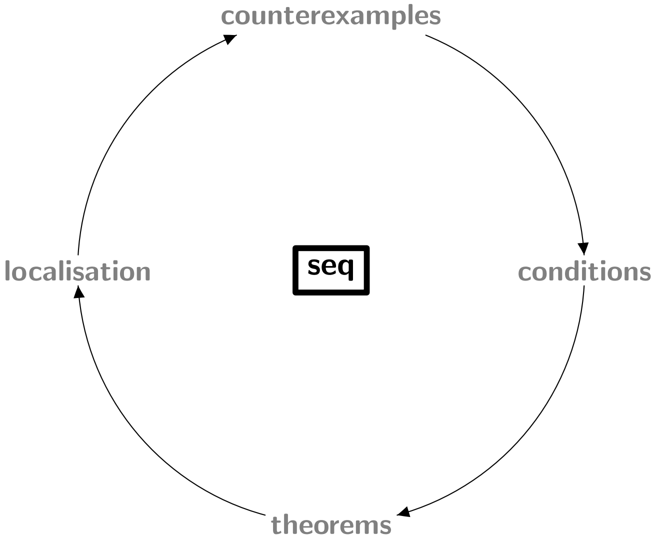


Progress for Selective Strictness



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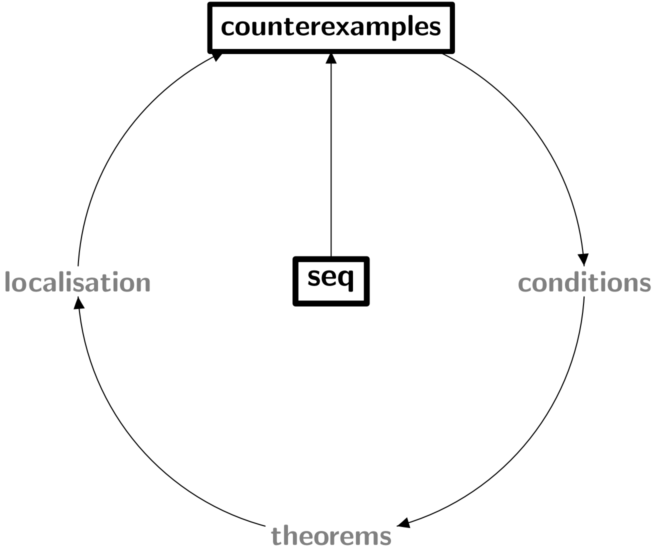

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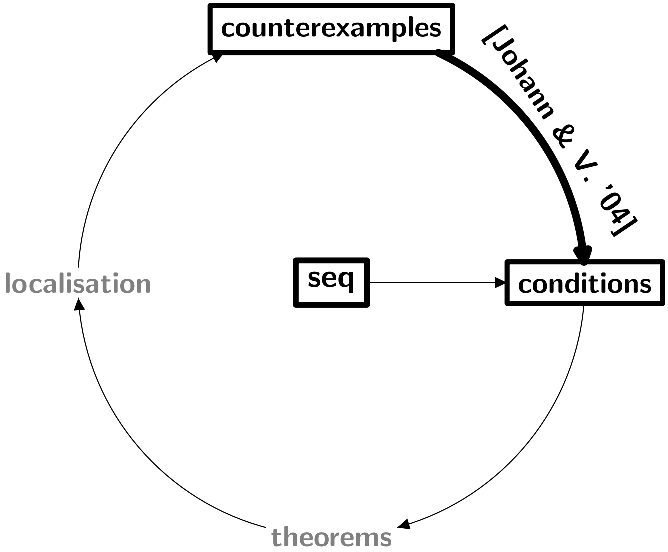

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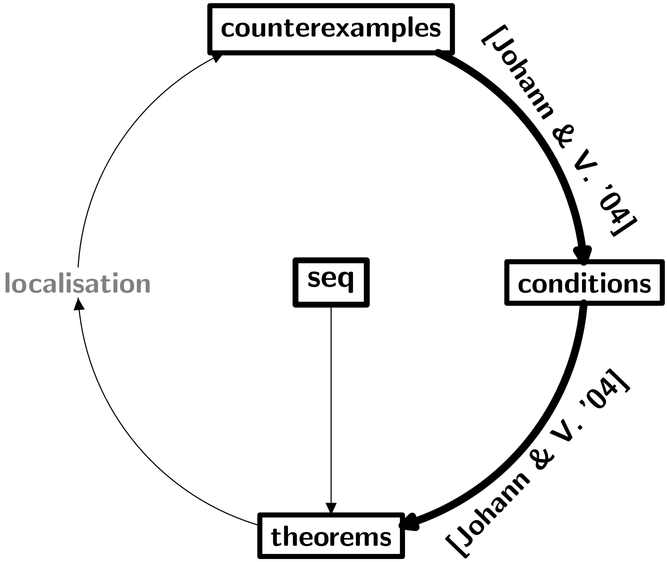

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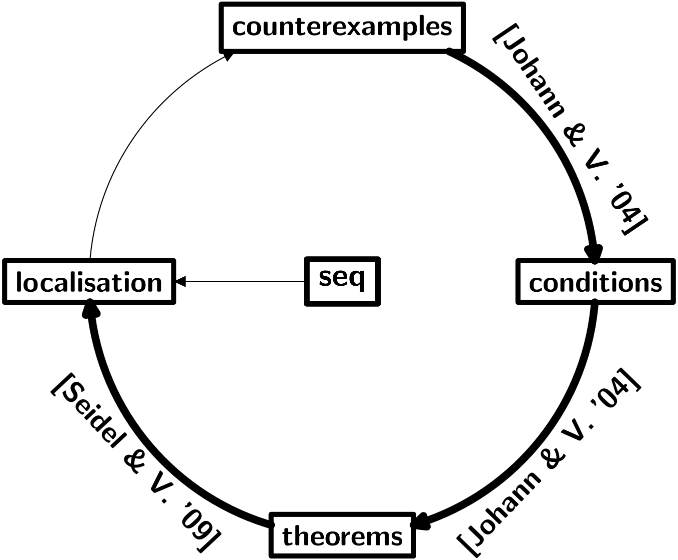
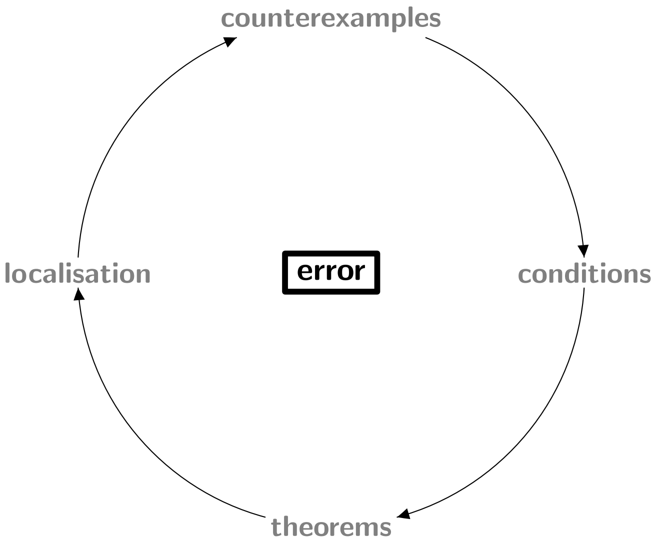


Progress for Imprecise Errors



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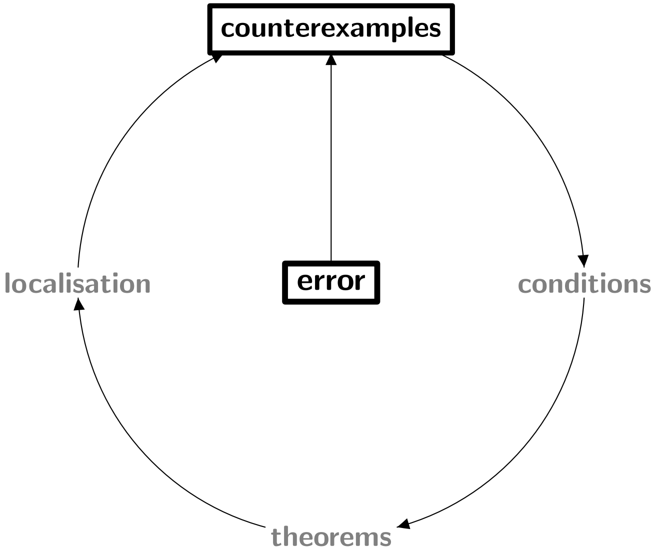

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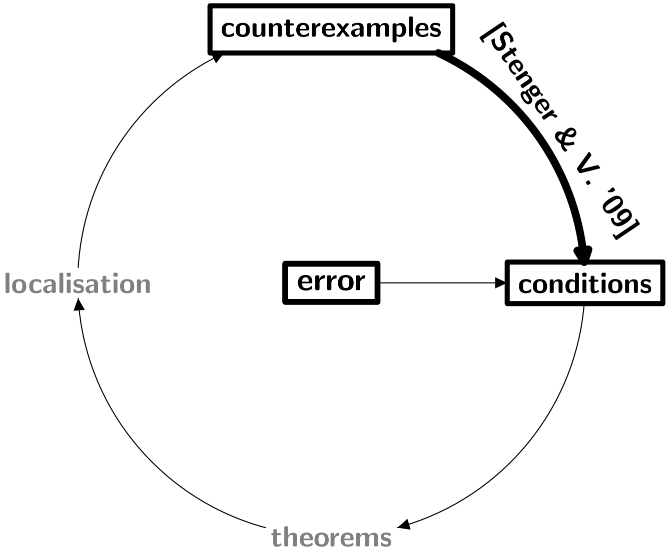

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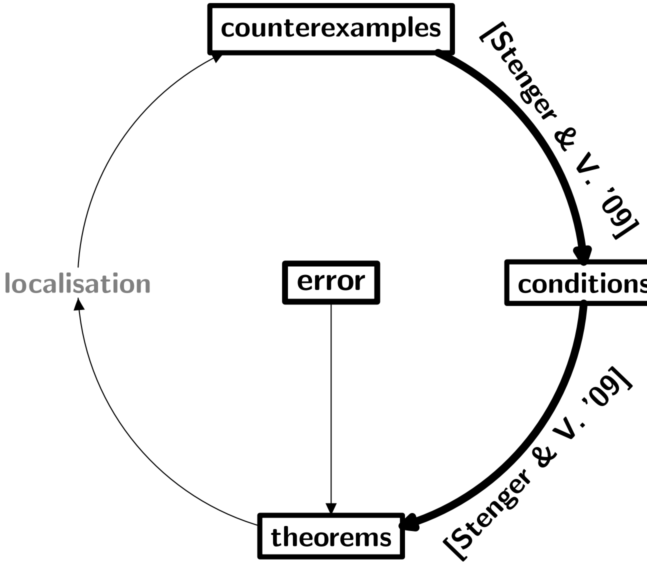
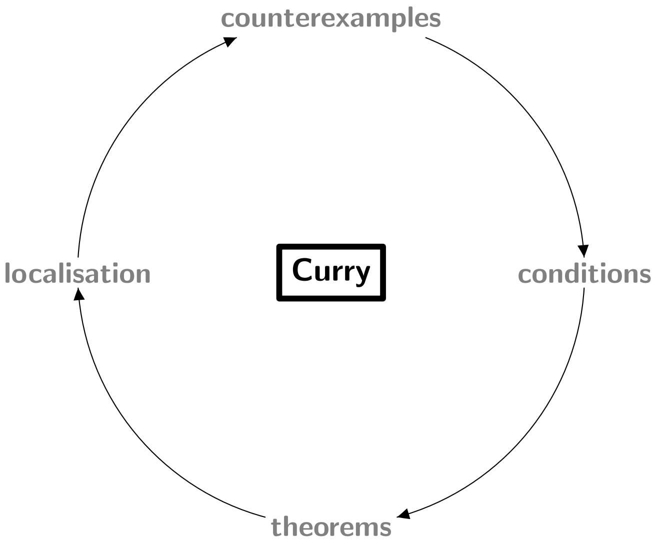


Progress for Functional Logic Programs



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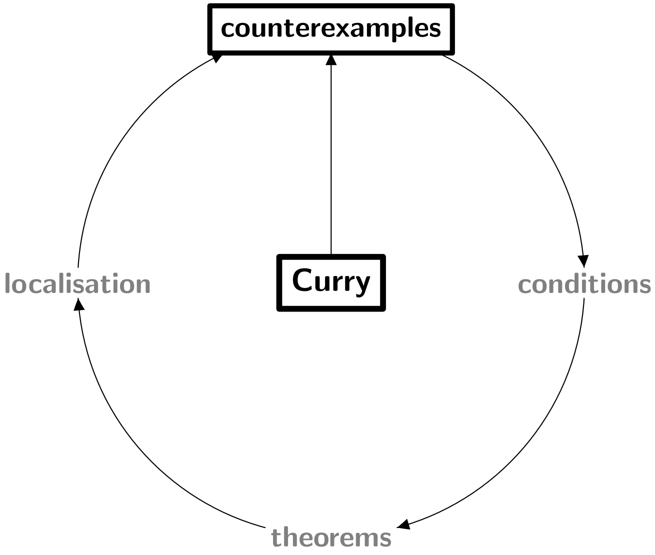

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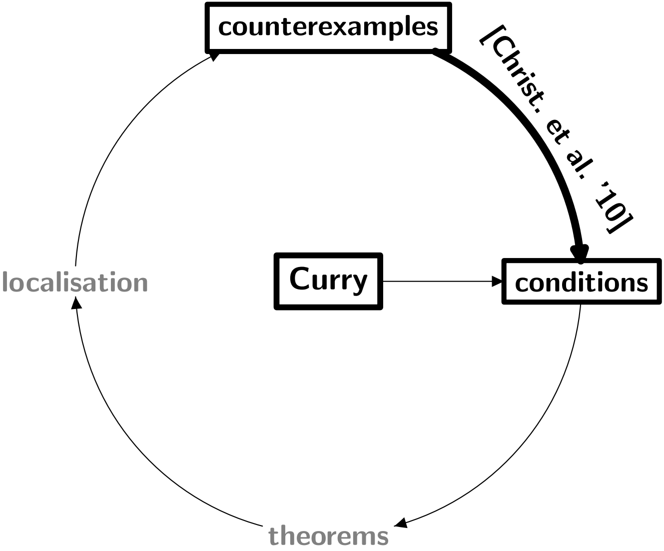
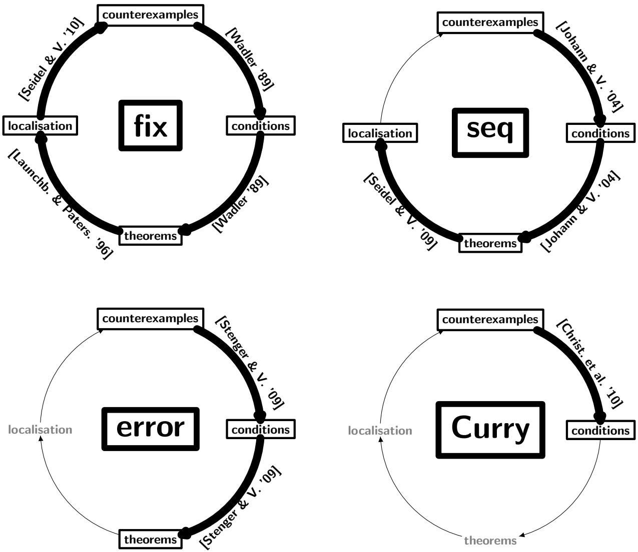
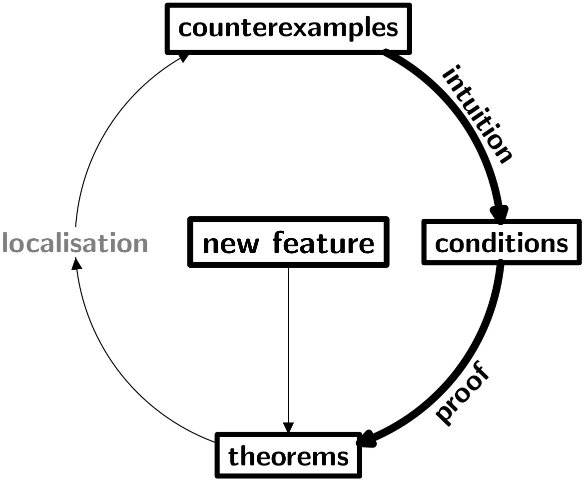


An Overview (and Challenges)



Impact on Applications



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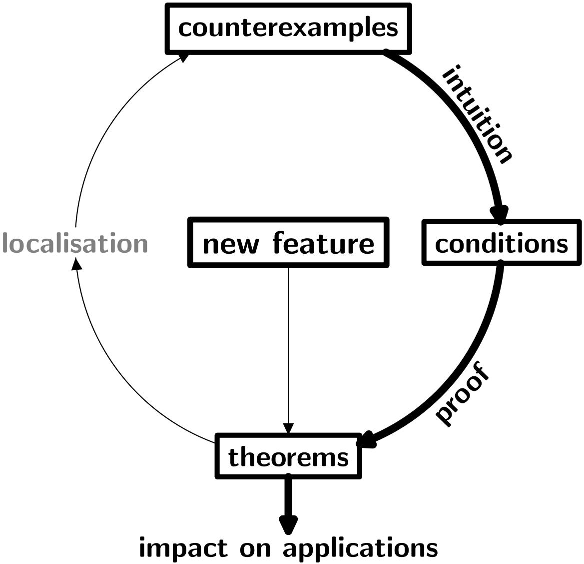
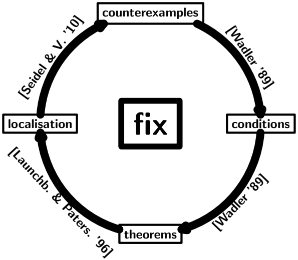
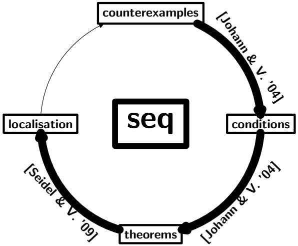
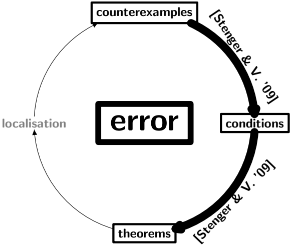


Specific Extensions and Specific Applications







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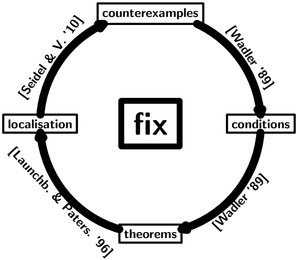

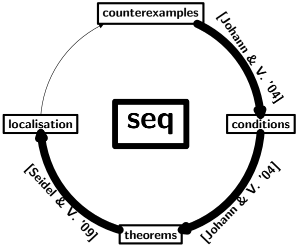

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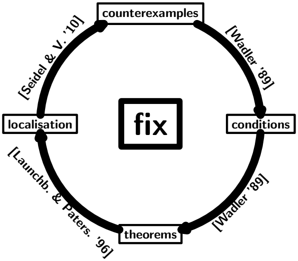

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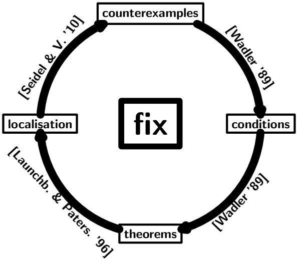

✓ Short Cut Fusion [Gill et al., FPCA'93]

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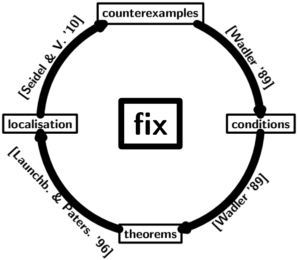
✓ Circular Short Cut Fusion [Fernandes et al., Haskell'07]

...

Specific Extensions and Specific Applications



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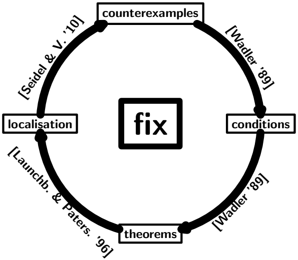

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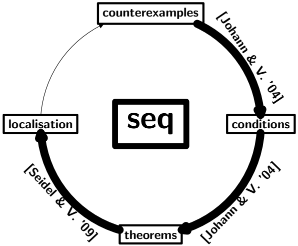
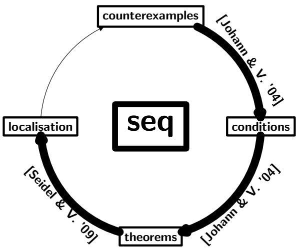
- ?/✓ Reasoning about invariants for monadic programs [V., ICFP'09]

Specific Extensions and Specific Applications



Specific Extensions and Specific Applications



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Conclusion

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