Much Ado about Two

A Pearl on Parallel Prefix Computation

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POPL'08

Parallel Prefix Computation

Given: inputs $x_1, \ldots, x_n$ and an associative operation $\oplus$

Task: compute the values $x_1, x_1 \oplus x_2, \ldots, x_1 \oplus x_2 \oplus \cdots \oplus x_n$

Parallel Prefix Computation

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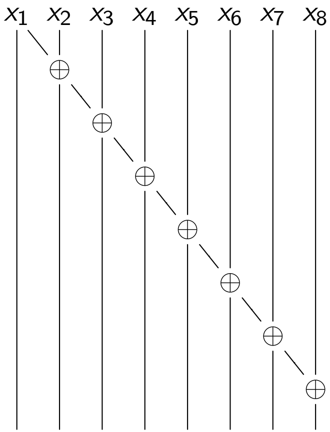
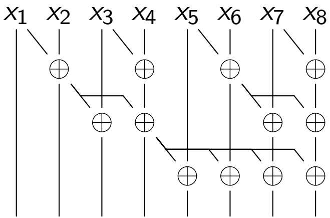
Solution:



Parallel Prefix Computation

Alternative:



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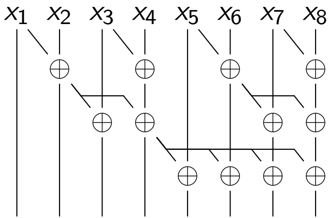
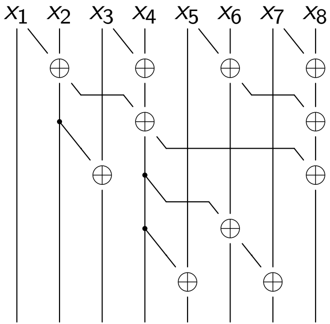
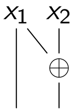


Or:



Or: ...

A la [Sklansky 1960]:



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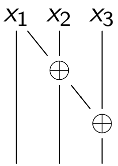

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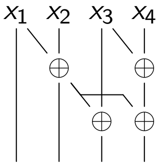

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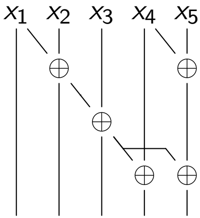

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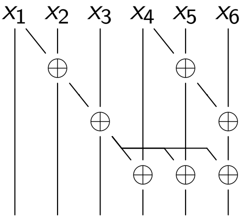

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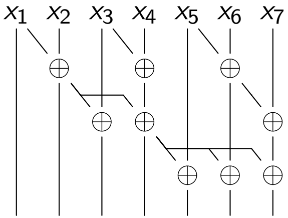

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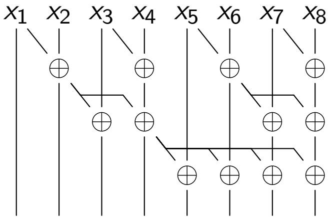

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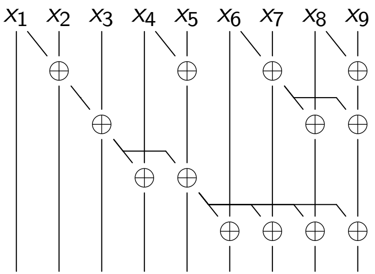

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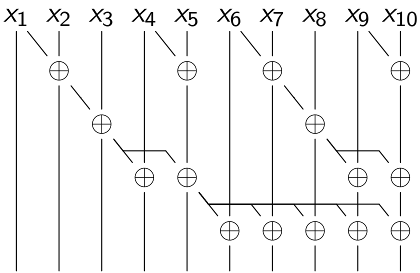

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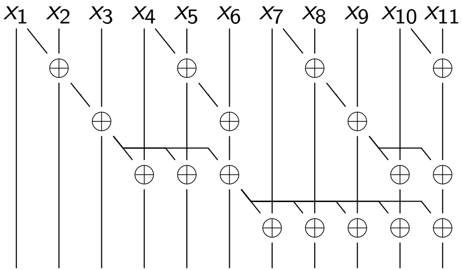

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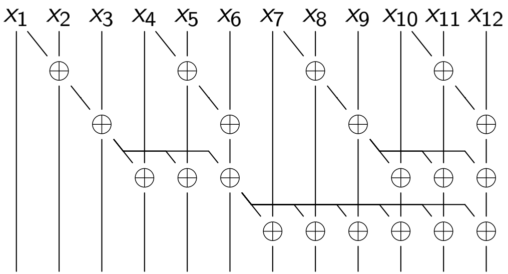

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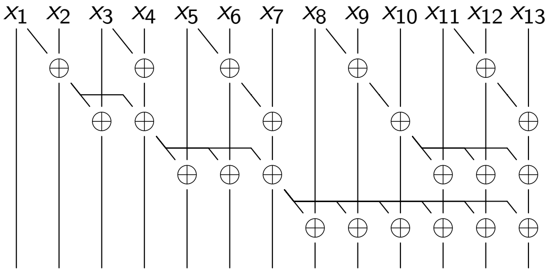

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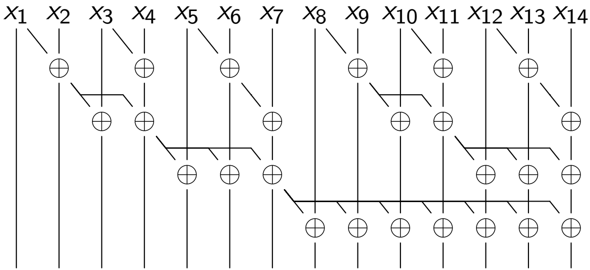

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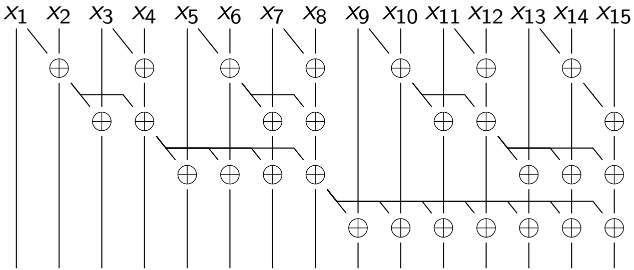

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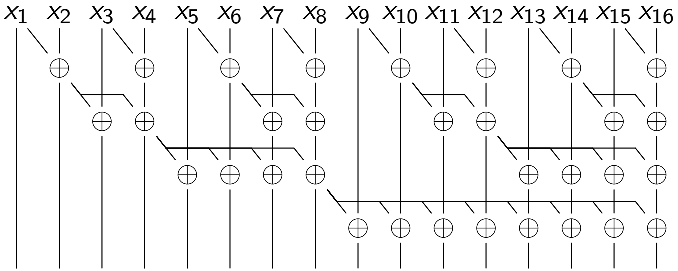

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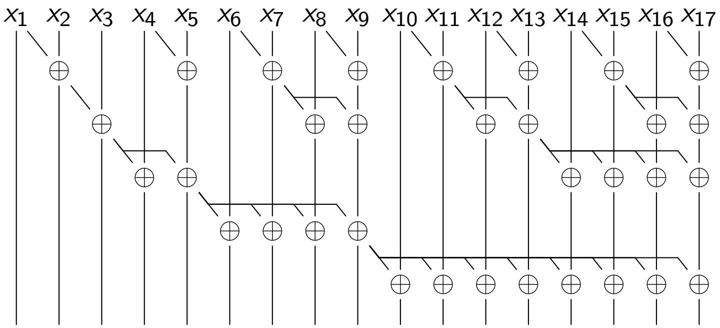

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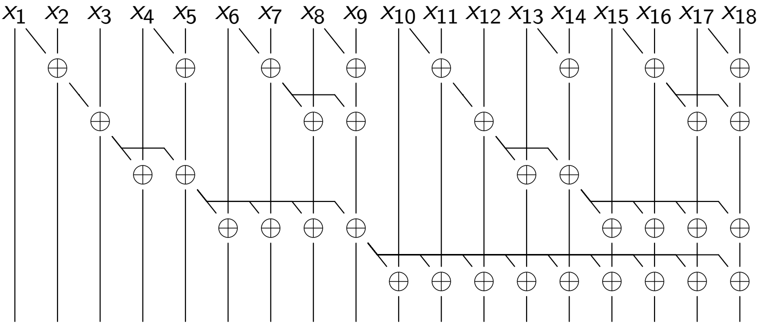

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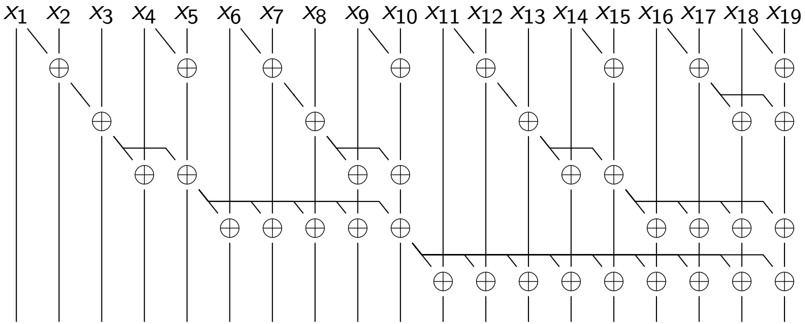

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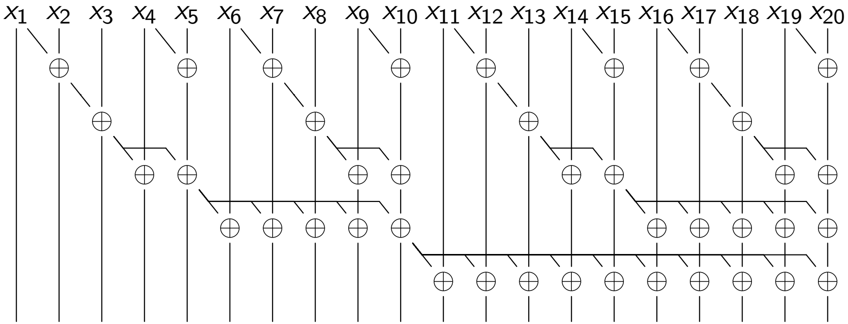

Sklansky's Method in Haskell

$$
\begin{aligned}
&\mathit{sklansky} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha] \\
&\mathit{sklansky}\ (\oplus)\ [x] = [x] \\
&\mathit{sklansky}\ (\oplus)\ \mathit{xs}\ = \mathit{us} +\!\!+ \mathit{vs} \\
&\qquad \textbf{where}\ t \qquad\quad\ = (\mathit{length}\ \mathit{xs} + 1)\ \text{`}\mathit{div}\text{`}\ 2 \\
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\end{array}
$$

Wanted: reasoning principles, verification techniques, systematic testing approach, ...

A Knuth-like 0-1-2-Principle

Given: $$\begin{aligned} & \mathit{serial} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha] \\ & \mathit{serial}\ (\oplus)\ (x : \mathit{xs}) = \mathit{go}\ x\ \mathit{xs} \\ & \qquad \textbf{where}\ \mathit{go}\ x\ [\,] \qquad\quad = [x] \\ & \qquad\qquad\quad\ \ \mathit{go}\ x\ (y : \mathit{ys}) = x : \mathit{go}\ (x \oplus y)\ \mathit{ys} \end{aligned}$$

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$$\textbf{data}\ \mathit{Three} = \mathit{Zero} \mid \mathit{One} \mid \mathit{Two}$$

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Theorem: If for every $\mathit{xs} :: [\mathit{Three}]$ and associative $(\oplus) :: \mathit{Three} \to \mathit{Three} \to \mathit{Three}$,

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then the same holds for every type τ, $\mathit{xs} :: [\tau]$, and associative $(\oplus) :: \tau \to \tau \to \tau$.

Why 0-1-2? And How?

A question: What can *candidate* $:: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$ do, given an operation $\oplus$ and input list $[x_1, \ldots, x_n]$?

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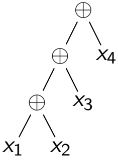
Among these expressions, there are good ones:



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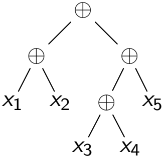

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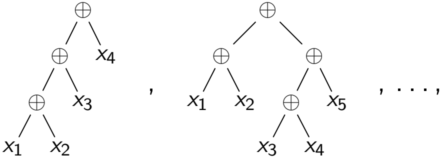

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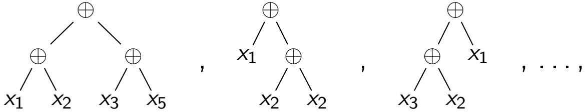

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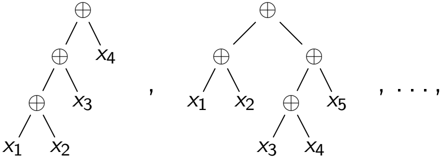

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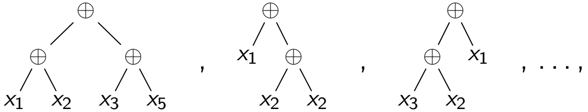
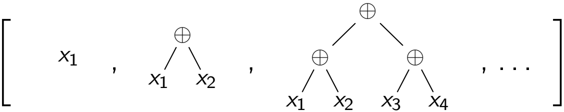


and ones in the wrong position:



That's How!

Let

$\oplus_1$	*Zero*	*One*	*Two*
Zero	*Zero*	*One*	*Two*
One	*One*	*Two*	*Two*
Two	*Two*	*Two*	*Two*

and

$\oplus_2$	*Zero*	*One*	*Two*
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If *candidate* ($\oplus_1$) is correct on each list of the form

$$[(Zero,)^* \; One \; (,Zero)^* \; (,Two)^*]$$

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Zero	*Zero*	*One*	*Two*
One	*One*	*Two*	*Two*
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If *candidate* $(\oplus_1)$ is correct on each list of the form

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and *candidate* $(\oplus_2)$ is correct on each list of the form

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then *candidate* is correct for associative $\oplus$ at arbitrary type.

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The Overall Proof

- To get going, uses relational parametricity [Reynolds 1983] to derive a free theorem from *candidate*'s type [Wadler 1989].
- Remaining proof largely done by program calculation. (But also a bit "by picture".)
- Formalisation available in Isabelle/HOL:

  S. Böhme.
  Much Ado about Two. Formal proof development.
  The Archive of Formal Proofs.
  `http://afp.sf.net/entries/MuchAdoAboutTwo.shtml`

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