Knuth's 0-1-Principle and Beyond

Janis Voigtländer

University of Bonn, Germany

(visiting NII Tokyo)

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The Sorting Problem

Task: Given a list and an order on the type of elements of this list, produce a sorted list (with same content)!

Example:

12	7	9	8	4	6

$\mapsto$

4	6	7	8	9	12

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$$\boxed{12 \mid 7 \mid 9 \mid 8 \mid 4 \mid 6} \mapsto \boxed{4 \mid 6 \mid 7 \mid 8 \mid 9 \mid 12}$$

Many Solutions:

- Quicksort
- Insertion Sort
- Merge Sort
- Bubble Sort
- ...

Quicksort

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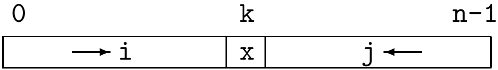

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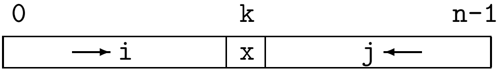

Example:

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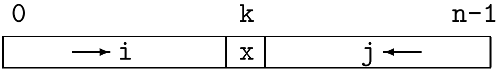

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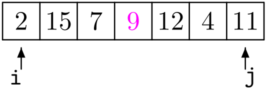

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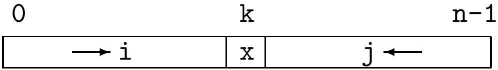

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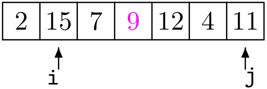

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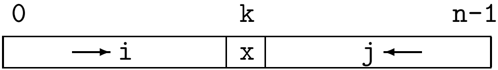

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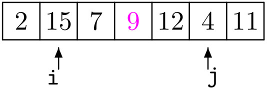

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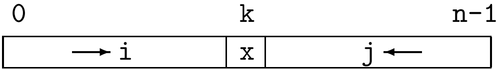

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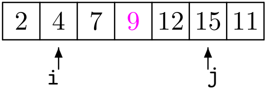

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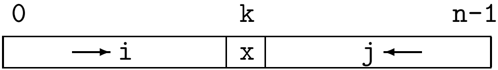

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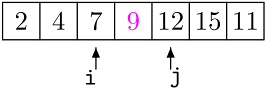

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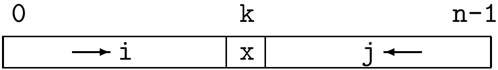

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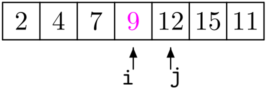

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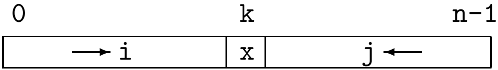

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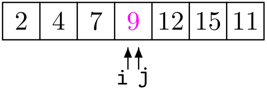

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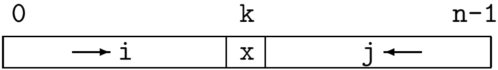

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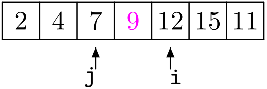

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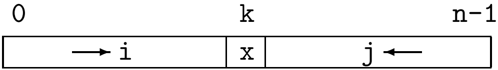

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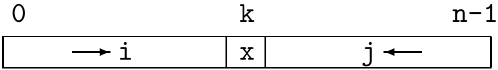

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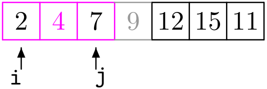

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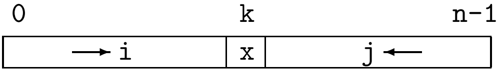

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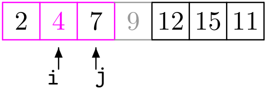

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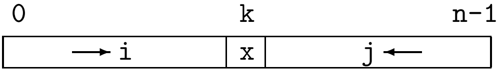

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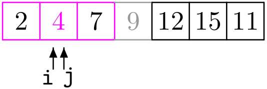

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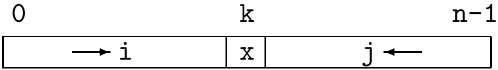

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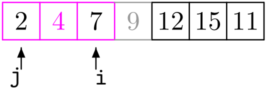

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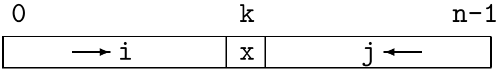

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Note:

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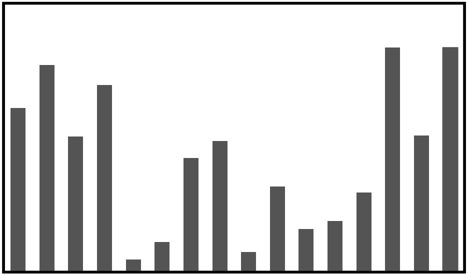

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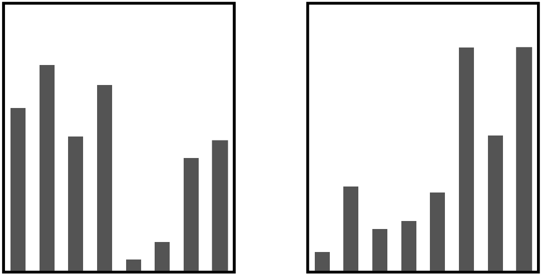

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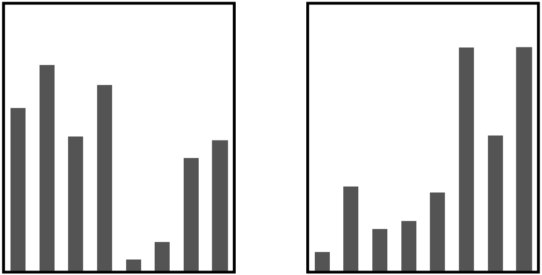

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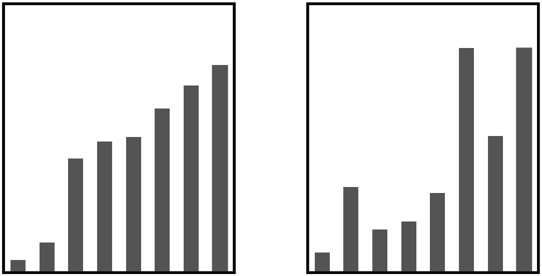

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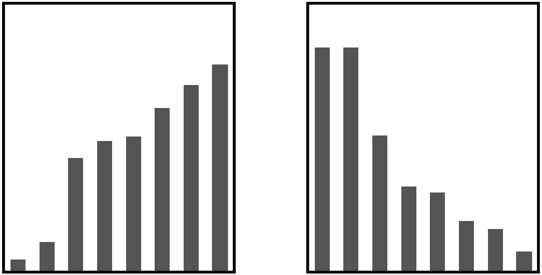

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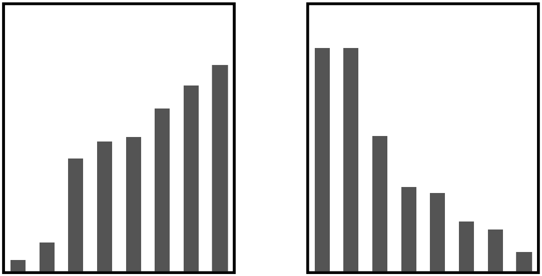

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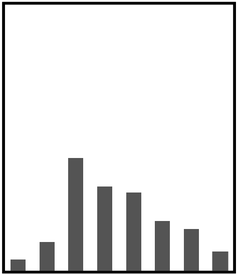

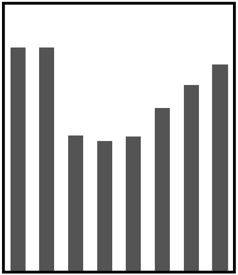

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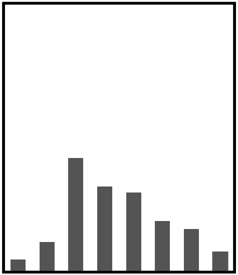

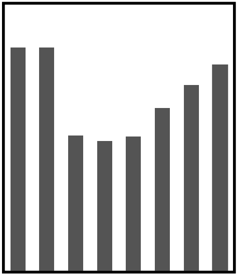

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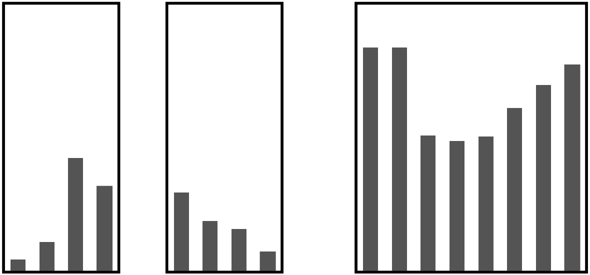

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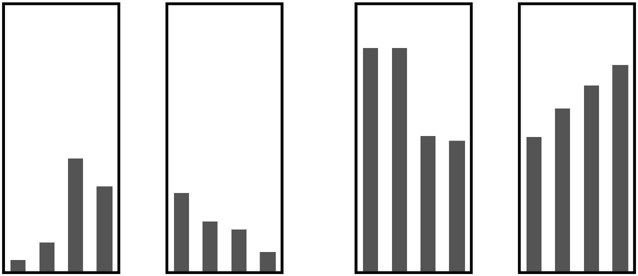

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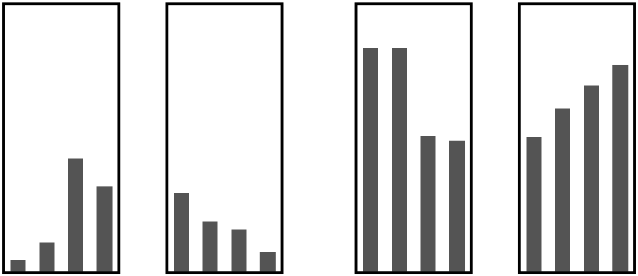

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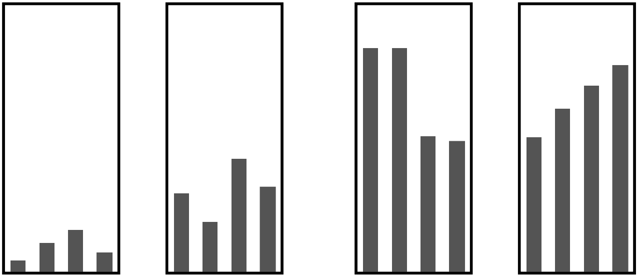

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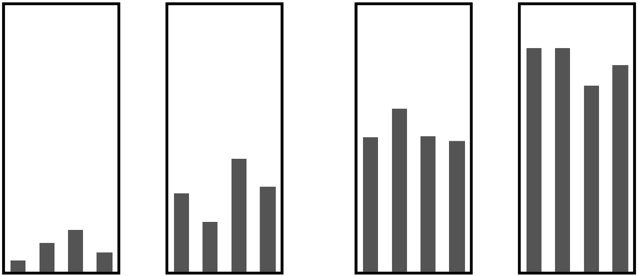

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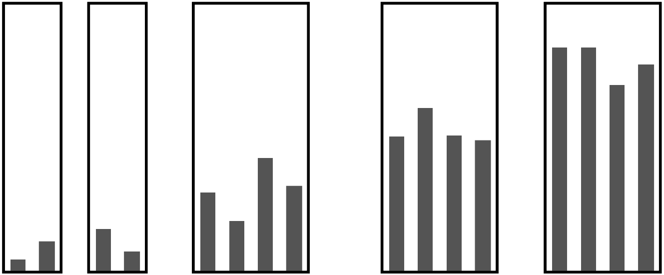

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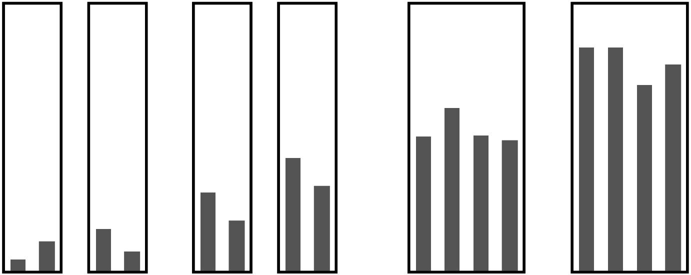

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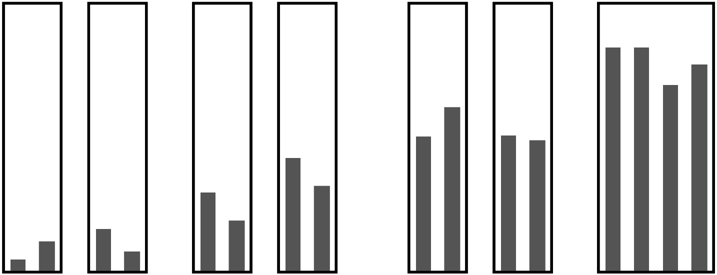

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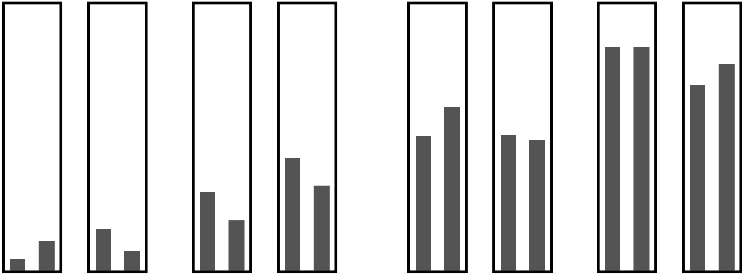

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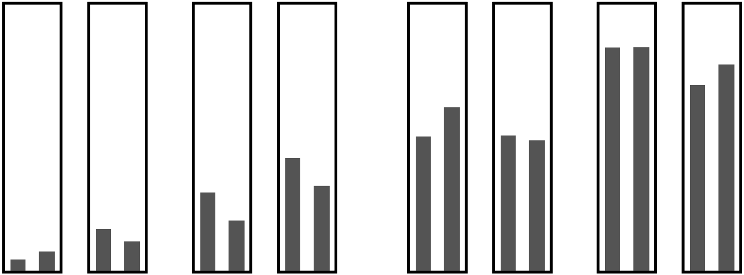

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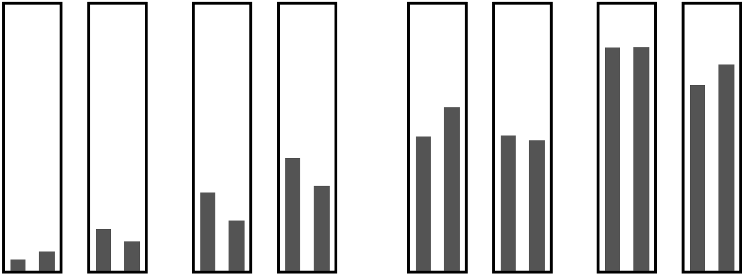

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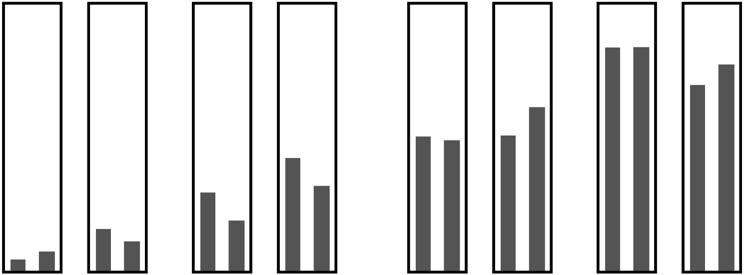

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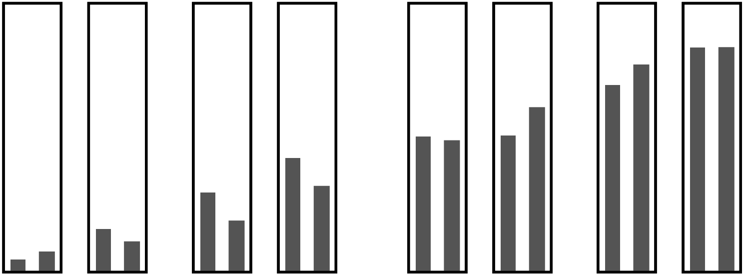

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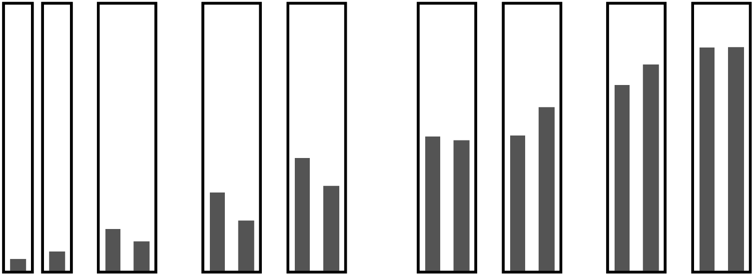

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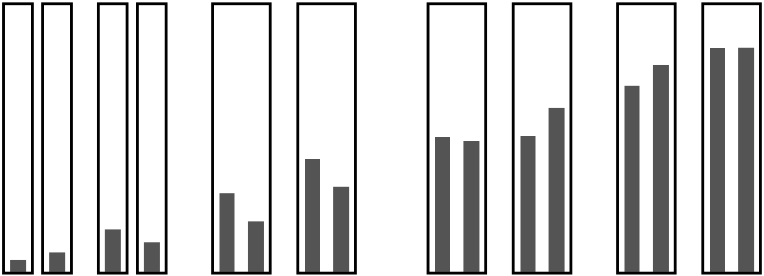

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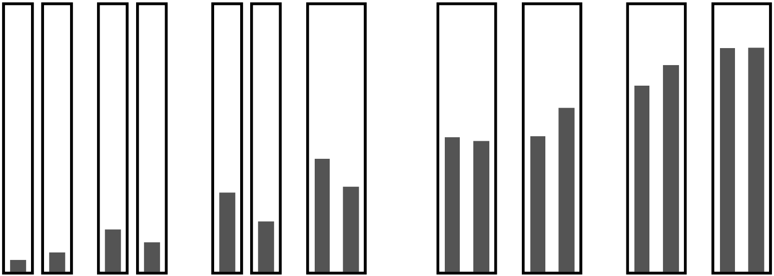

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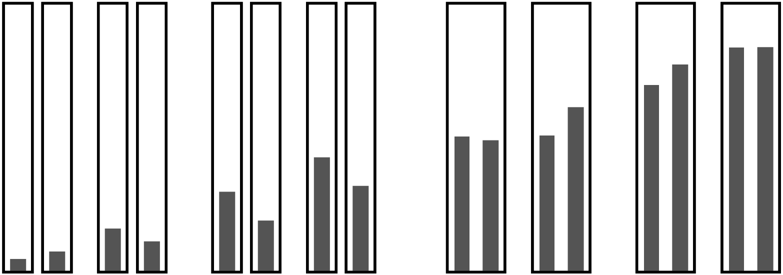

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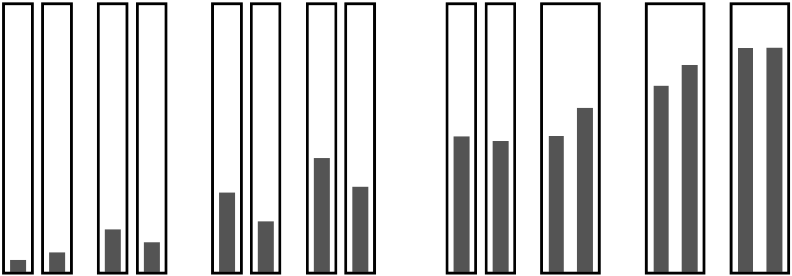

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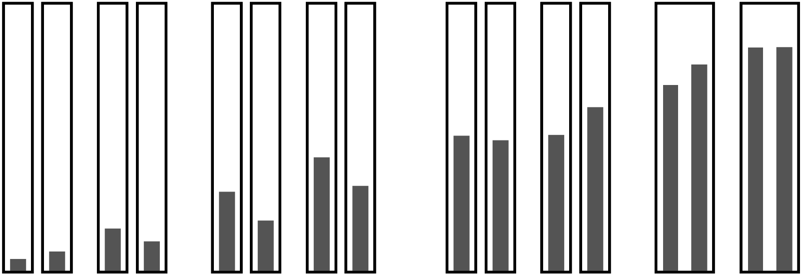

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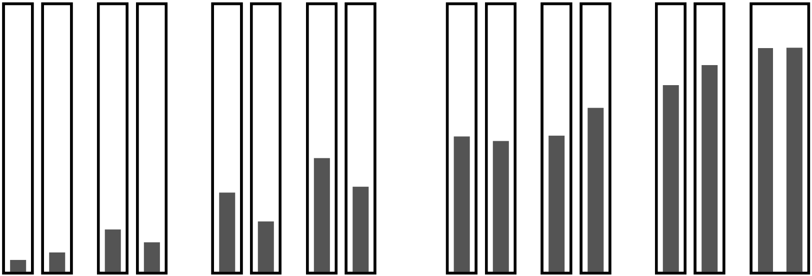

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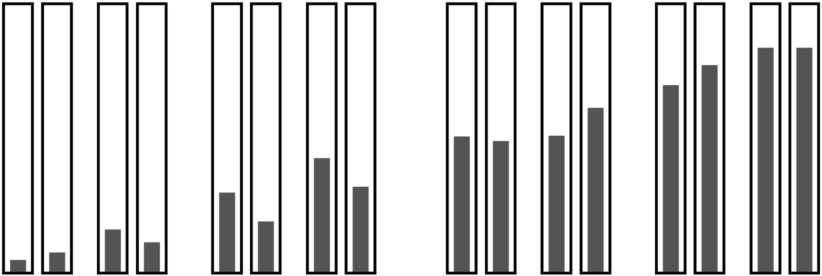

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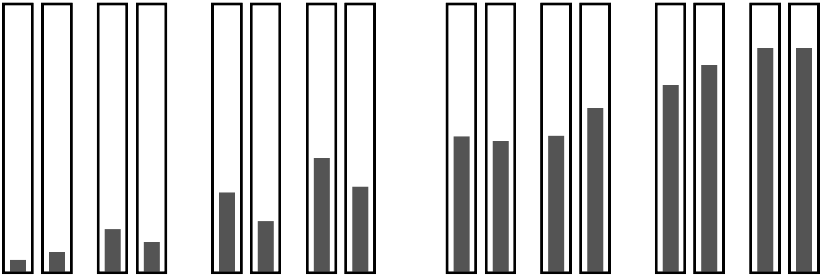

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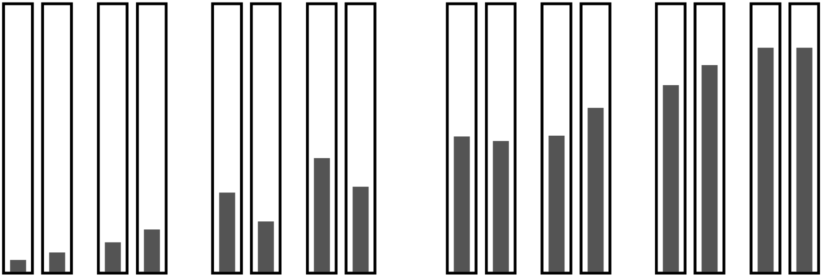

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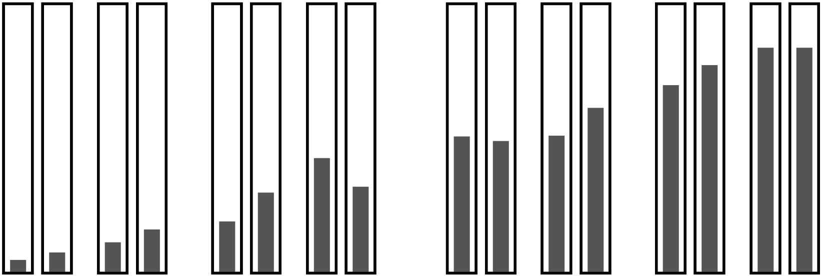

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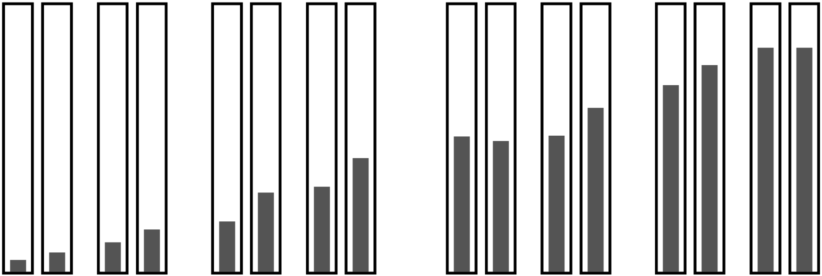

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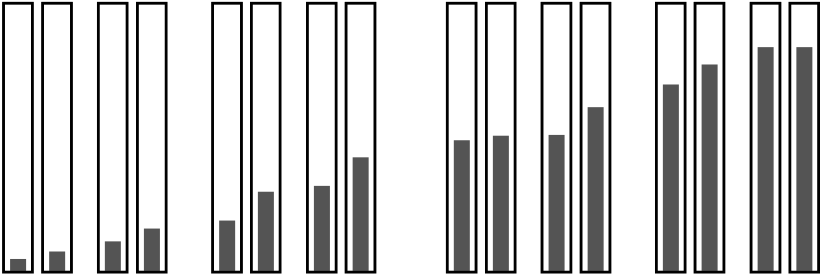

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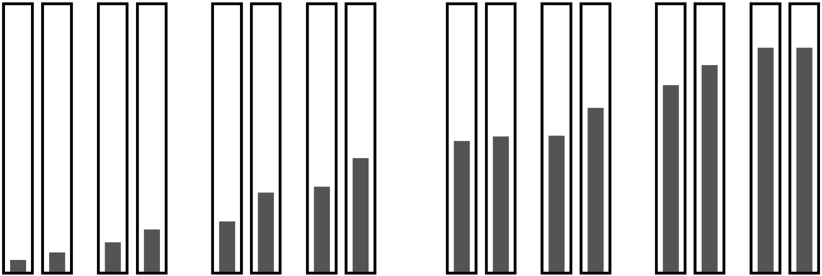

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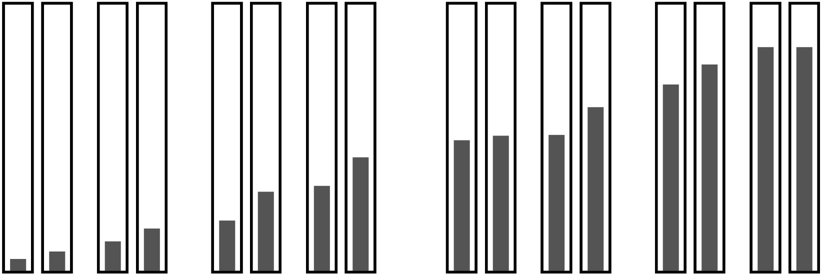

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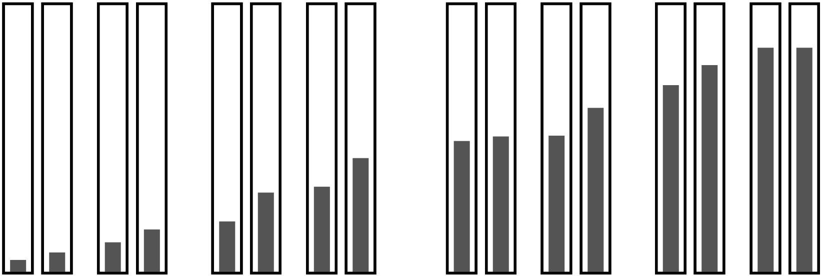

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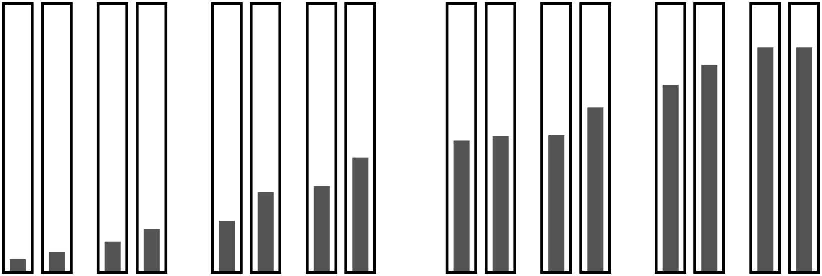

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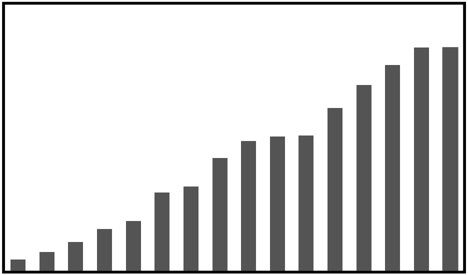

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- correctness is not obvious

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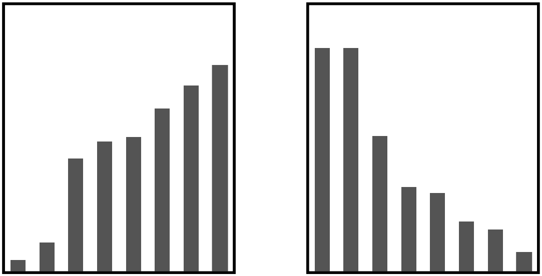

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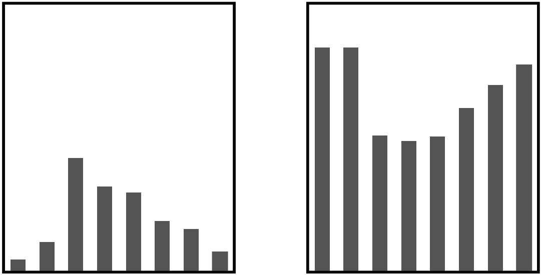

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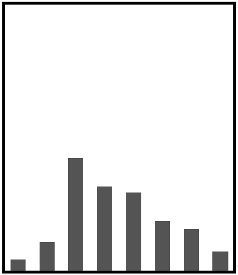

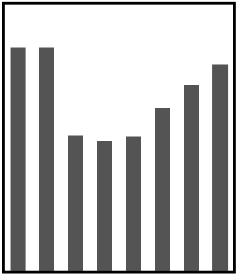

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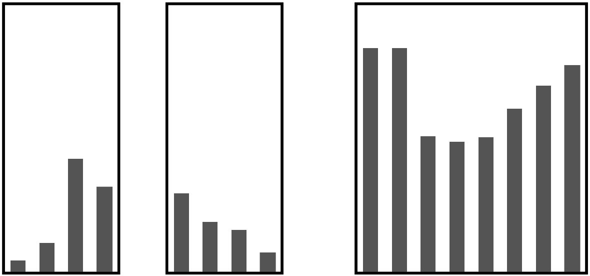

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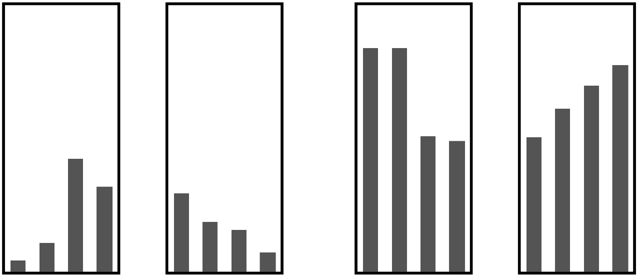

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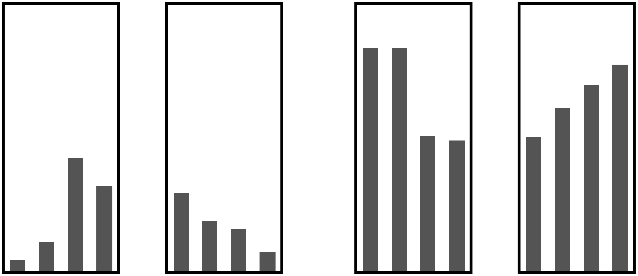

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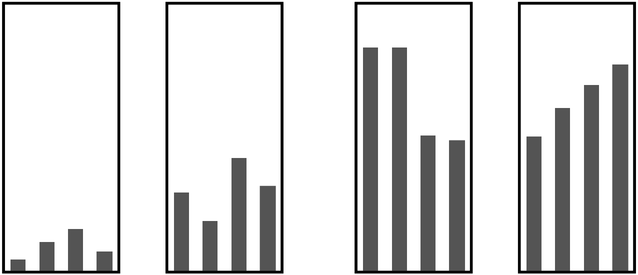

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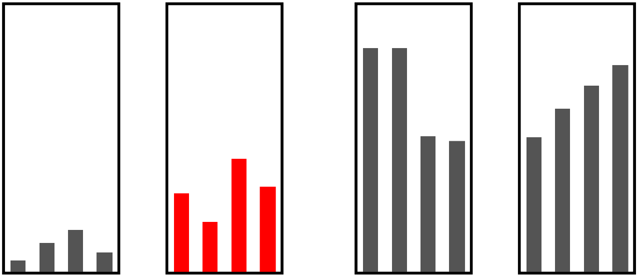

Knuth's 0-1-Principle [Knuth 1973]

Informally: If a comparison-swap algorithm sorts Booleans correctly, it sorts integers correctly as well.

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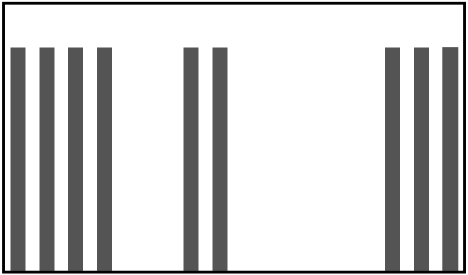

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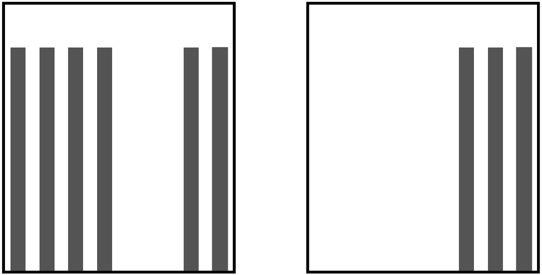

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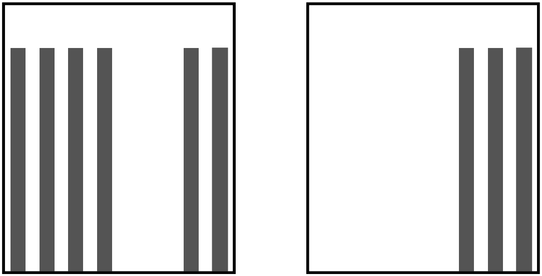

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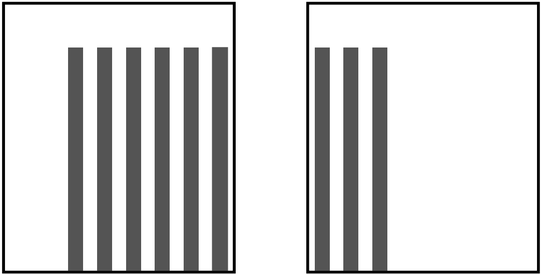

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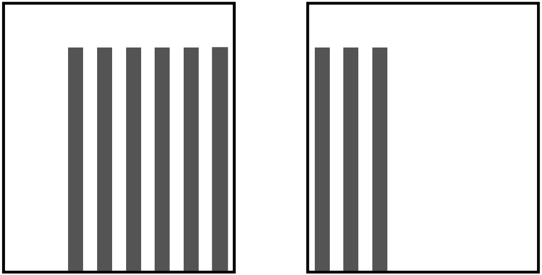

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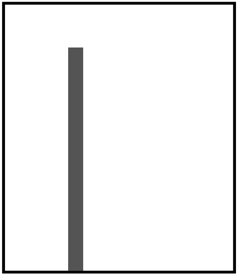

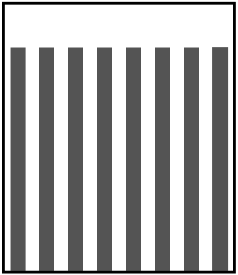

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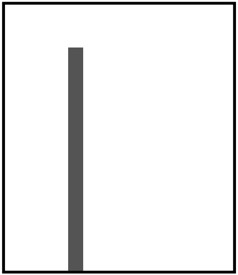

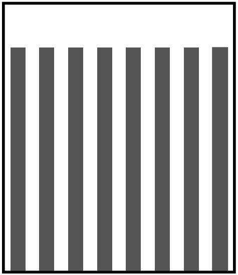

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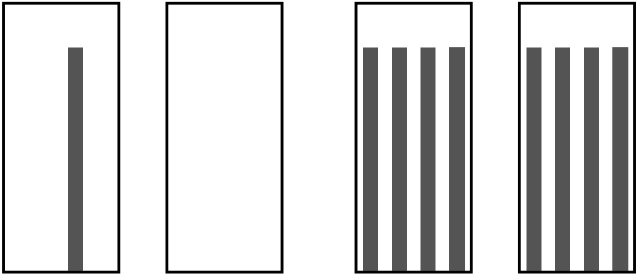

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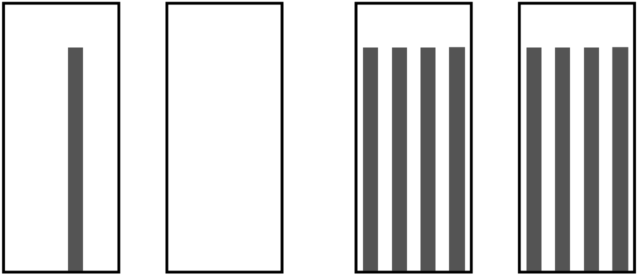

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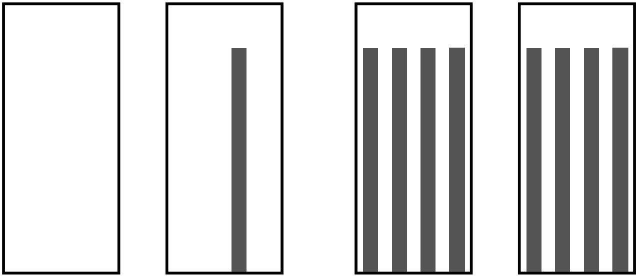

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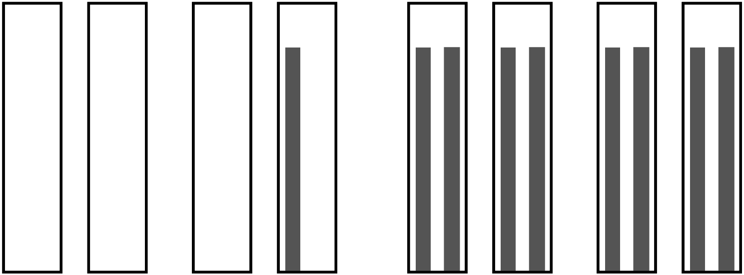

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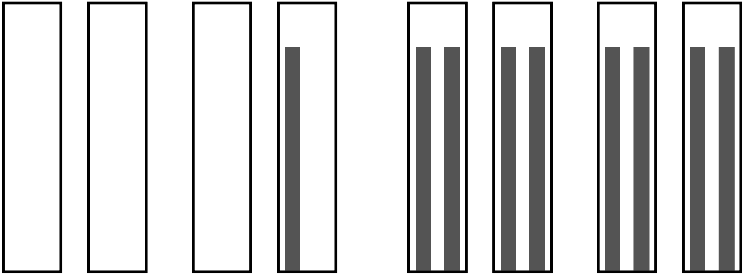

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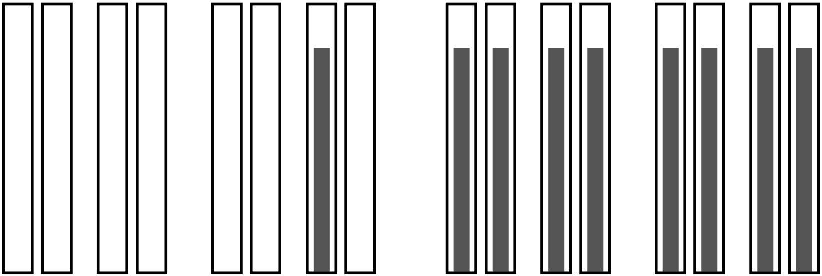

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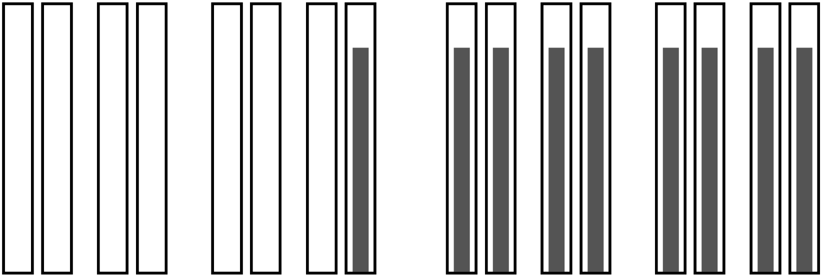

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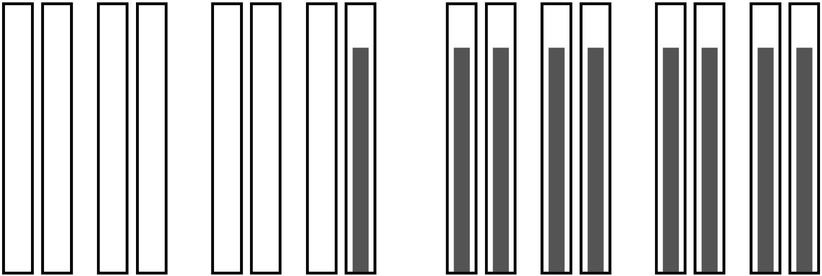

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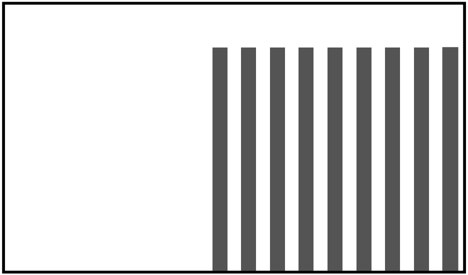

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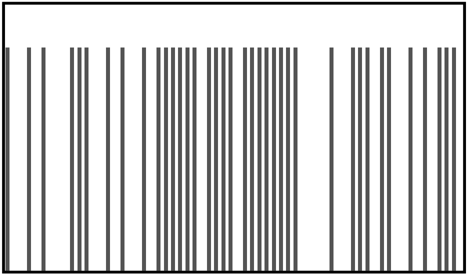

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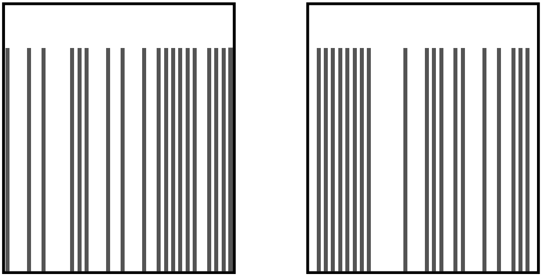

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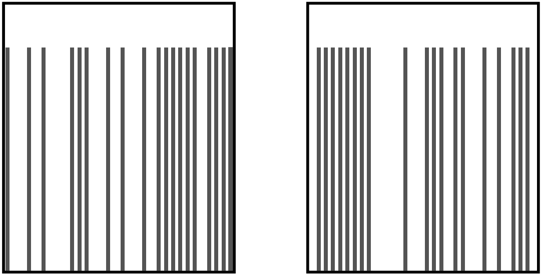

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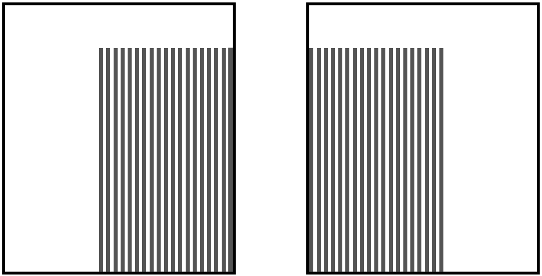

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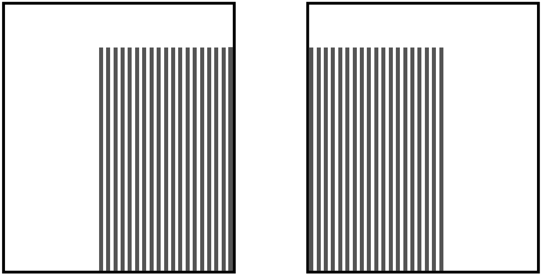

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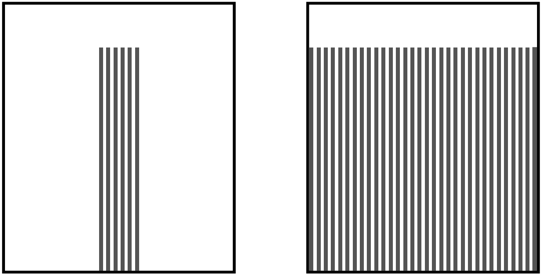

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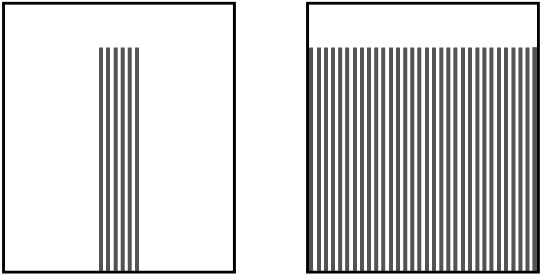

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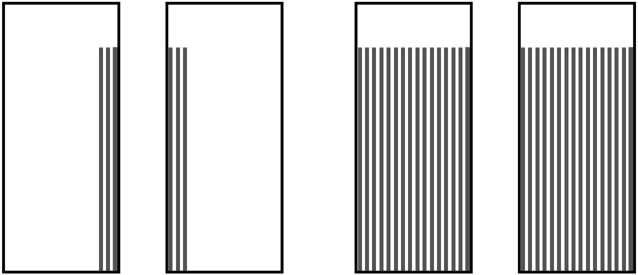

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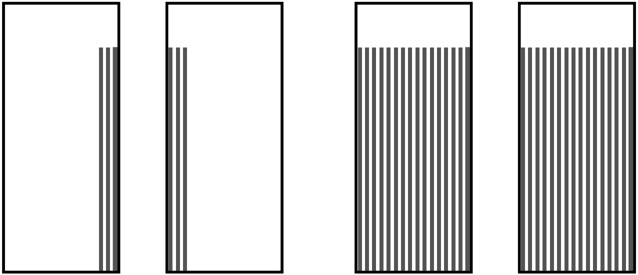

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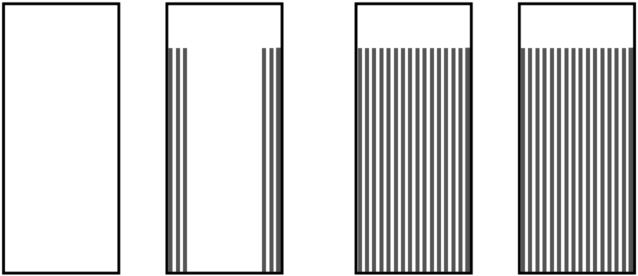

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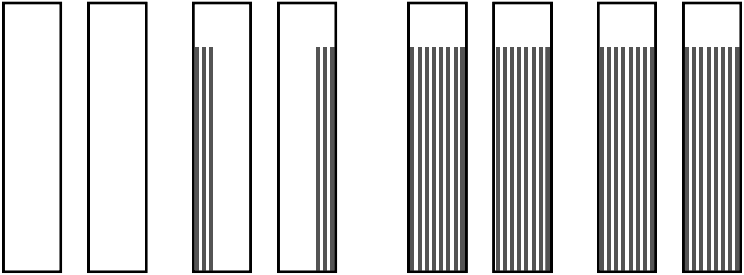

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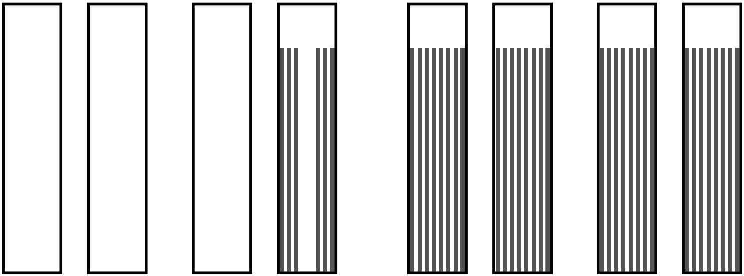

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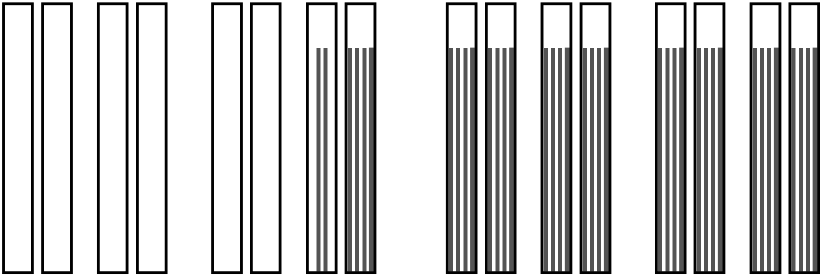

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Informally: If a comparison-swap algorithm sorts Booleans correctly, it sorts integers correctly as well.

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If for every $xs :: [\mathsf{Bool}]$, $\texttt{sort}\ g\ xs$ gives the correct result, then for every $xs :: [\mathsf{Int}]$, $\texttt{sort}\ f\ xs$ gives the correct result.

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Free Theorems [Reynolds 1983, Wadler 1989]

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Input: `sort::((a,a)->(a,a))->[a]->[a]`

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Input:
```
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```

Output:
```
forall t1,t2 in TYPES, h::t1->t2.
 forall f::(t1,t1)->(t1,t1).
  forall g::(t2,t2)->(t2,t2).
   (forall (x,y) in lift_{(,)}(h,h).
     (f x,g y) in lift_{(,)}(h,h))
   ==> (forall xs::[t1].
         map h (sort f xs) = sort g (map h xs))
```

```
lift_{(,)}(h,h)
 = {((x1,x2),(y1,y2)) | (h x1 = y1)
                         && (h x2 = y2)}
```

```
map h []     = []
map h (x:xs) = (h x):(map h xs)
```

More Specific (and Intuitive)

For every $\texttt{sort} :: ((\alpha, \alpha) \to (\alpha, \alpha)) \to [\alpha] \to [\alpha]$,
$f :: (\mathsf{Int}, \mathsf{Int}) \to (\mathsf{Int}, \mathsf{Int})$, $g :: (\mathsf{Bool}, \mathsf{Bool}) \to (\mathsf{Bool}, \mathsf{Bool})$, and
$h :: \mathsf{Int} \to \mathsf{Bool}$:

$$\begin{tikzcd}
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If f and g are as defined before, then the precondition is fulfilled for any h of the form $h\ x = n < x$ for some $n :: \mathsf{Int}$.

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Proof of “P on [Bool] implies P on [Int]”

Recall: $P(xs, ys) :=$ *xs* and *ys* contain the same elements in the same multiplicity

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Given: $\forall xs :: [\text{Bool}], ys = \texttt{sort}\ g\ xs.\ P(xs, ys)$

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Assume there exist $us :: [\text{Int}]$ and $vs = \texttt{sort}\ f\ us$ with $\neg P(us, vs)$. Then there is a smallest integer n such that the multiplicities of n in us and vs are not the same.

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Given: $\forall xs :: [\text{Bool}], ys = \texttt{sort}\ g\ xs.\ P(xs, ys)$

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Assume there exist $us :: [\text{Int}]$ and $vs = \texttt{sort}\ f\ us$ with $\neg P(us, vs)$. Then there is a smallest integer n such that the multiplicities of n in us and vs are not the same. Then for $h\ x = n < x$ the multiplicities of False in ($\texttt{map}\ h\ us$) and ($\texttt{map}\ h\ vs$) are different.

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Knuth's 0-1-Principle [Knuth 1973]

Informally: If a comparison-swap algorithm sorts Booleans correctly, it sorts integers correctly as well.

Formally: Use Haskell. Let

$$\texttt{sort} :: ((\alpha, \alpha) \to (\alpha, \alpha)) \to [\alpha] \to [\alpha]$$

$$f :: (\mathsf{Int}, \mathsf{Int}) \to (\mathsf{Int}, \mathsf{Int})$$
$$f\ (x, y) = \mathit{if}\ x > y\ \mathit{then}\ (y, x)\ \mathit{else}\ (x, y)$$

$$g :: (\mathsf{Bool}, \mathsf{Bool}) \to (\mathsf{Bool}, \mathsf{Bool})$$
$$g\ (x, y) = (x\ \&\&\ y, x \mathbin{||} y)$$

If for every $xs :: [\mathsf{Bool}]$, $\texttt{sort}\ g\ xs$ gives the correct result, then for every $xs :: [\mathsf{Int}]$, $\texttt{sort}\ f\ xs$ gives the correct result.

And Beyond?

- Knuth's 0-1-Principle allows to reduce algorithm correctness, for comparison-swap sorting, for inputs over an infinite range to correctness over a finite range of values.

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- Free theorems allow for a particularly elegant proof of this principle. (This was not my idea: [Day et al. 1999]!)
- Can we do something similar for other algorithm classes?
- Good candidates: algorithms parametrised over some operation, like $\mathit{cswap} :: (\alpha, \alpha) \to (\alpha, \alpha)$ in the case of sorting.

Parallel Prefix Computation

Given: inputs $x_1, \ldots, x_n$ and an associative operation $\oplus$

Task: compute the values $x_1, x_1 \oplus x_2, \ldots, x_1 \oplus x_2 \oplus \cdots \oplus x_n$

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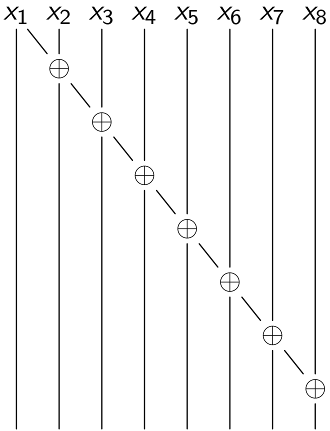
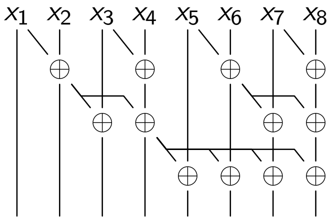
Solution:



Parallel Prefix Computation

Alternative:



Parallel Prefix Computation

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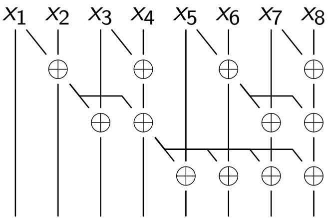
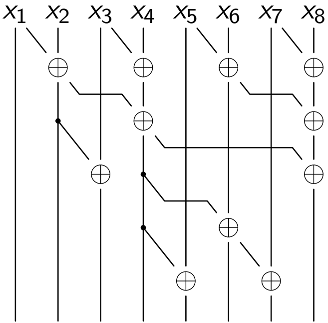


Or:



Or: ...

In Haskell

Functions of type:

$$\texttt{scanl1} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$$

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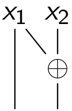
For example, à la [Sklansky 1960]:

$$\begin{aligned}
&\texttt{sklansky} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha] \\
&\texttt{sklansky}\ (\oplus)\ [x] = [x] \\
&\texttt{sklansky}\ (\oplus)\ xs\ \ = us +\!\!+\, vs \\
&\qquad \textbf{where}\ t \qquad\quad = ((\texttt{length}\ xs) + 1)\ `\texttt{div}`\ 2 \\
&\qquad\qquad\quad (ys, zs) = \texttt{splitAt}\ t\ xs \\
&\qquad\qquad\quad us \qquad = \texttt{sklansky}\ (\oplus)\ ys \\
&\qquad\qquad\quad vs \qquad = [(\texttt{last}\ us) \oplus v \mid v \leftarrow \texttt{sklansky}\ (\oplus)\ zs]
\end{aligned}$$

Sklansky's Method Visualised



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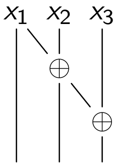

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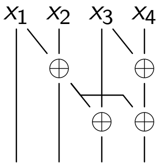

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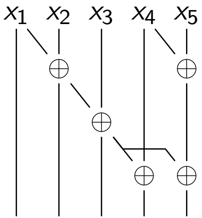

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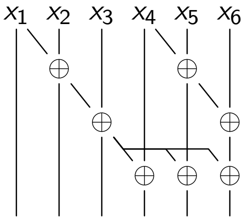

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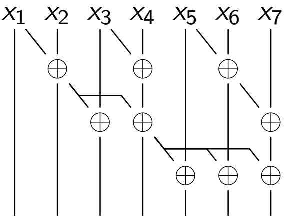

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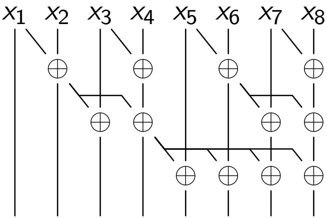

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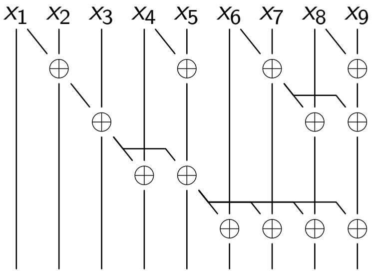

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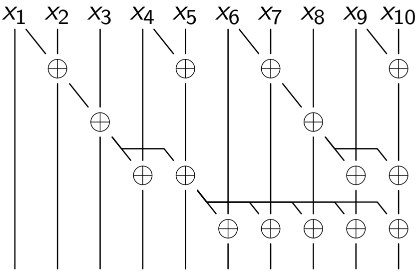

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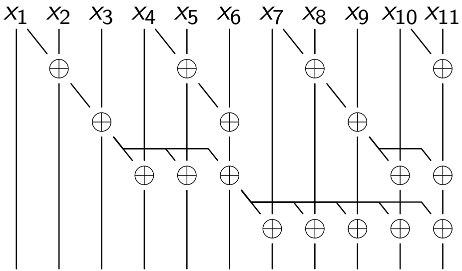

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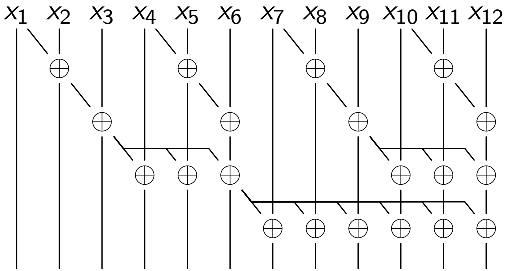

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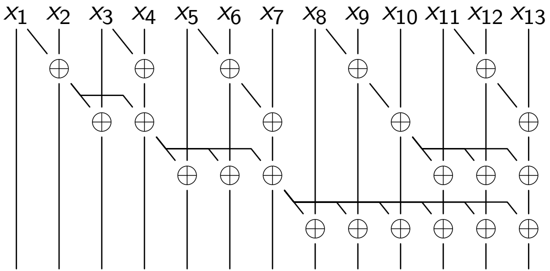

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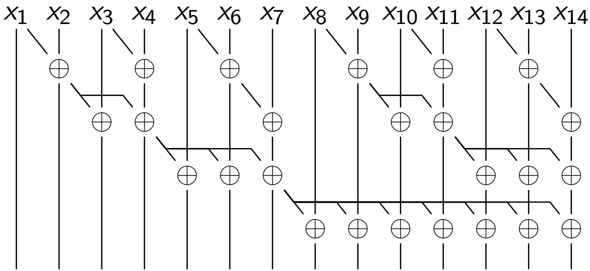

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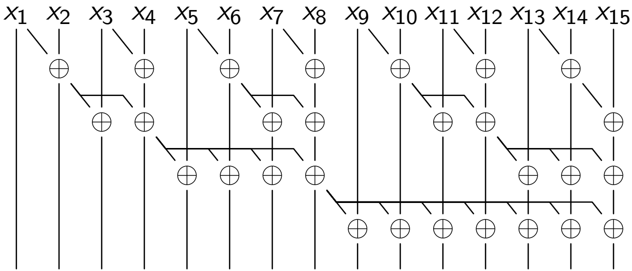

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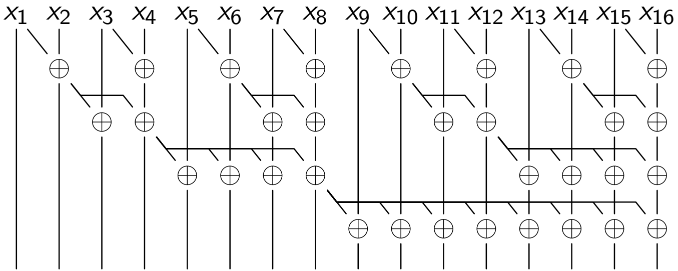

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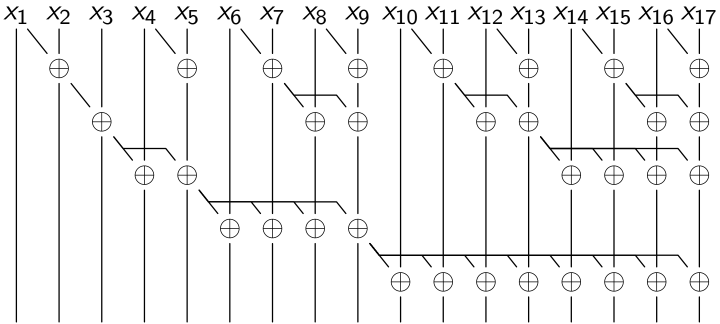

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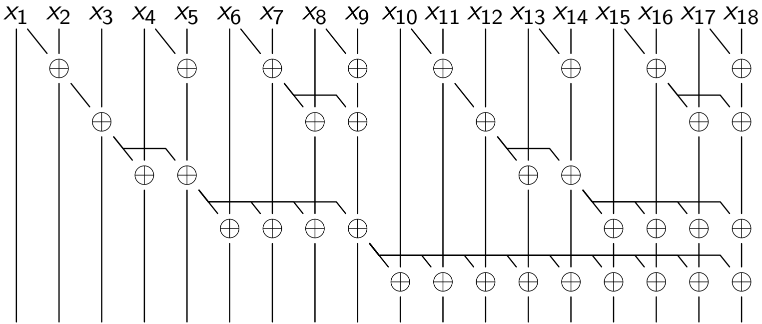

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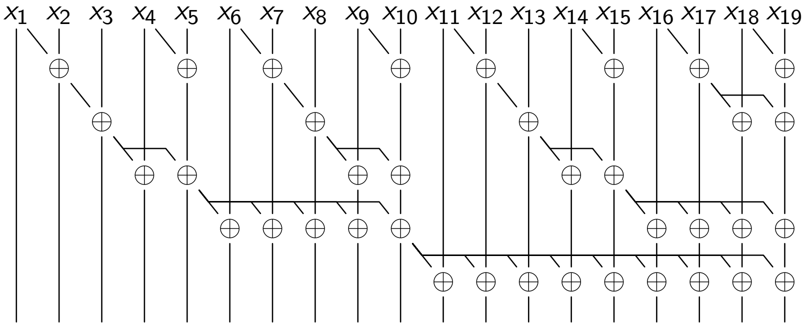

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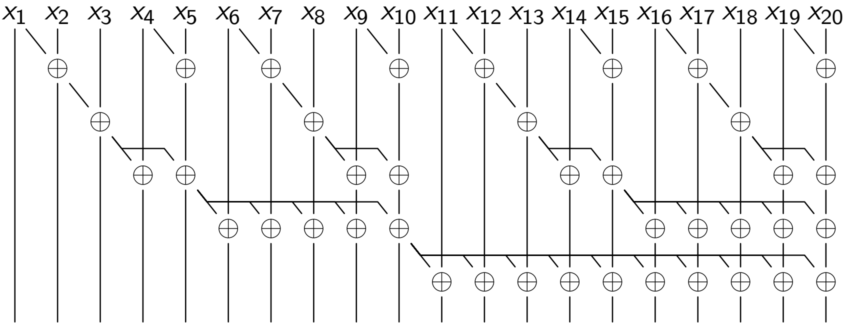
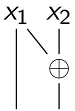


Or, à la [Brent & Kung 1980] (code follows [Sheeran 2007])

$$
\begin{aligned}
&\texttt{brentKung} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha] \\
&\texttt{brentKung}\ (\oplus)\ [x] = [x] \\
&\texttt{brentKung}\ (\oplus)\ xs\ = odds\ (riffle\ (par\ (unriffle\ (evens\ xs)))) \\
&\quad \textbf{where}\ evens\ [\,] \qquad\qquad = [\,] \\
&\qquad\qquad evens\ [x] \qquad\qquad = [x] \\
&\qquad\qquad evens\ (x : y : zs) = [x, x \oplus y] +\!\!+\ evens\ zs \\
&\qquad\qquad unriffle\ [\,] \qquad\qquad = ([\,],[\,]) \\
&\qquad\qquad unriffle\ [x] \qquad\qquad = ([x],[\,]) \\
&\qquad\qquad unriffle\ (x : y : zs) = (x : xs, y : ys) \\
&\qquad\qquad\quad \textbf{where}\ (xs, ys) = unriffle\ zs \\
&\qquad\qquad par\ (xs, ys) = (xs, \texttt{brentKung}\ (\oplus)\ ys) \\
&\qquad\qquad riffle\ ([\,],[\,]) \qquad\qquad = [\,] \\
&\qquad\qquad riffle\ ([x],[\,]) \qquad\qquad = [x] \\
&\qquad\qquad riffle\ (x : xs, y : ys) = x : y : riffle\ (xs, ys) \\
&\qquad\qquad odds\ (x : xs) = x : evens\ xs
\end{aligned}
$$

Brent & Kung's Method Visualised



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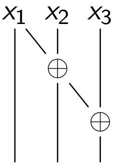

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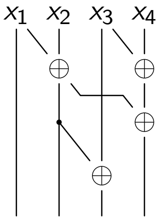

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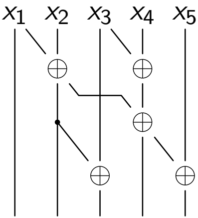

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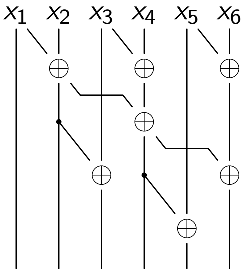

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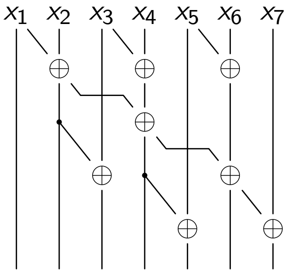

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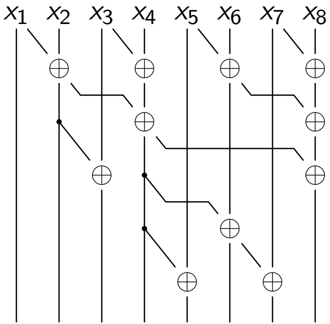

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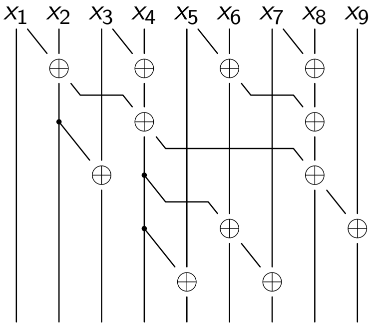

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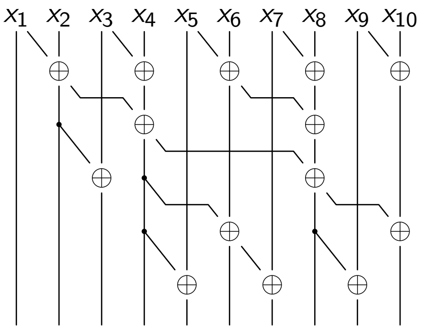

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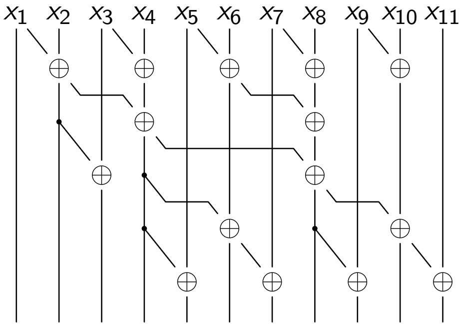

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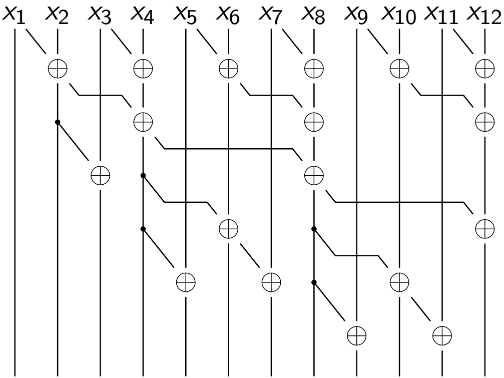

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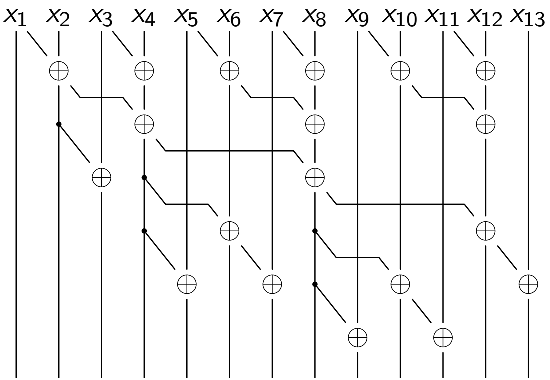

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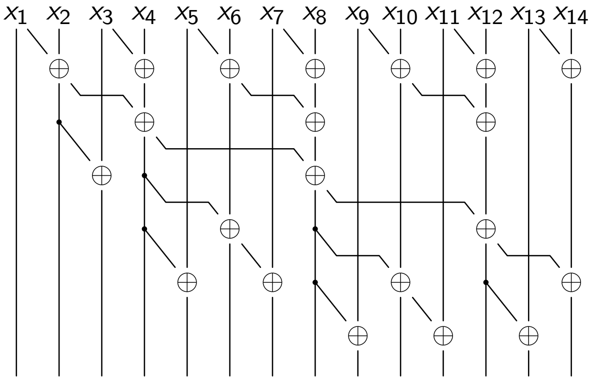

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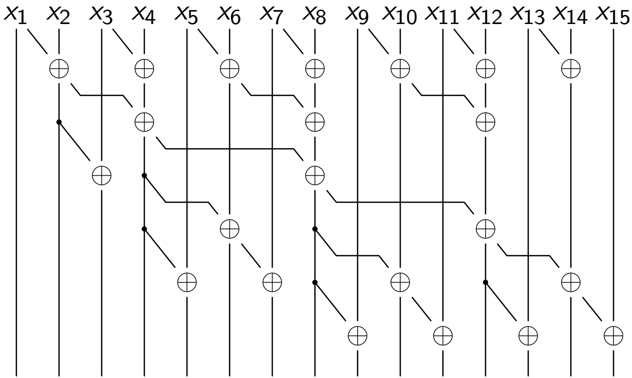

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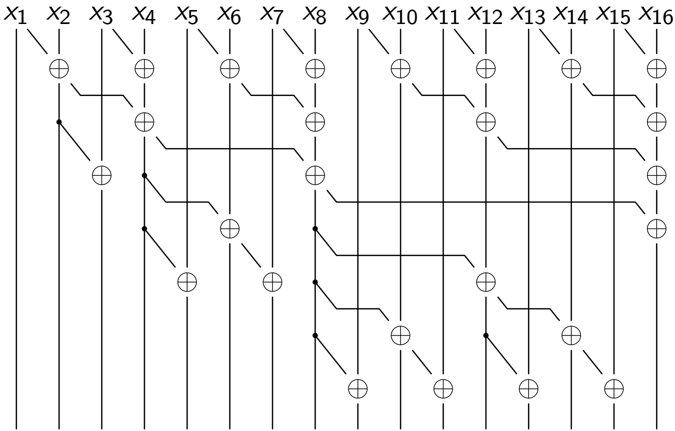

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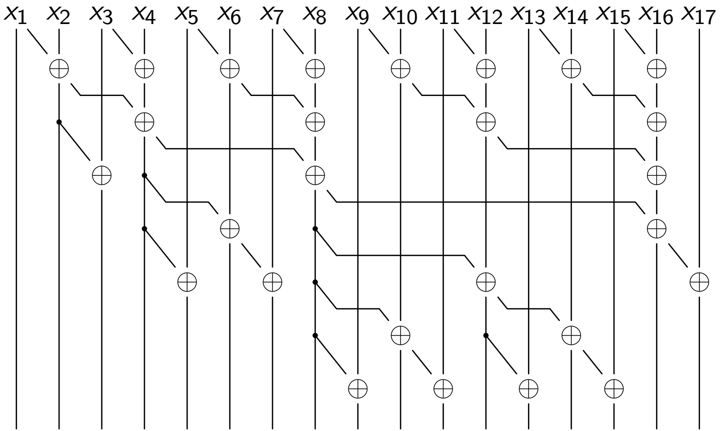

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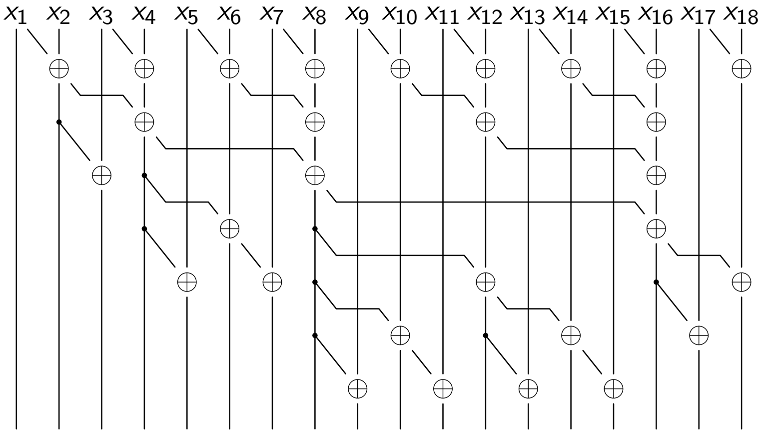

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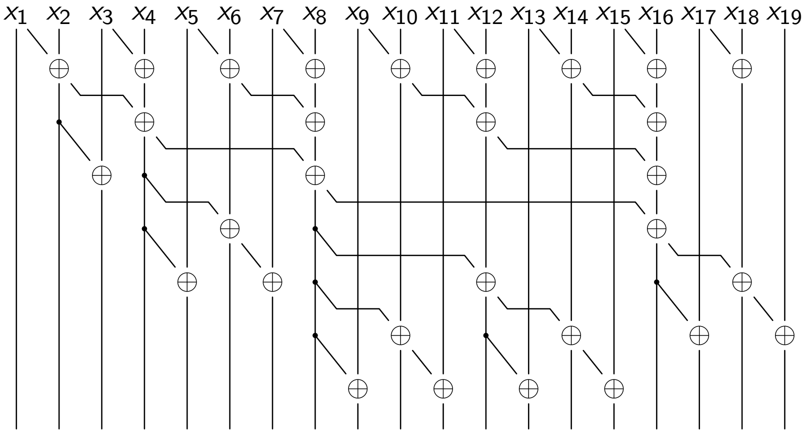

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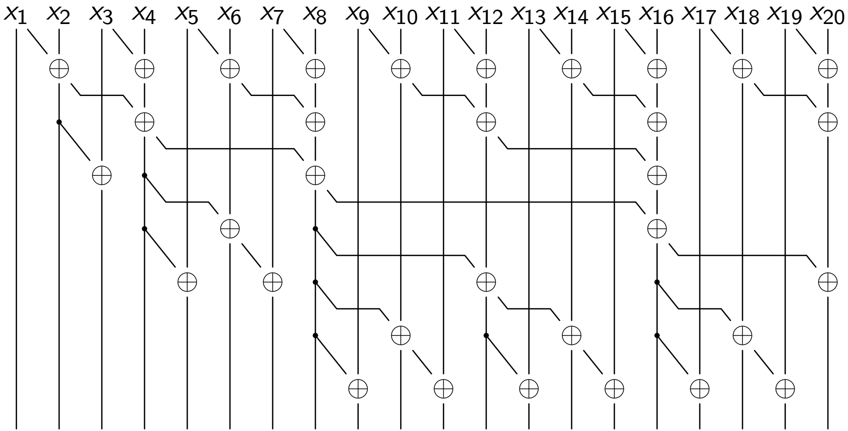

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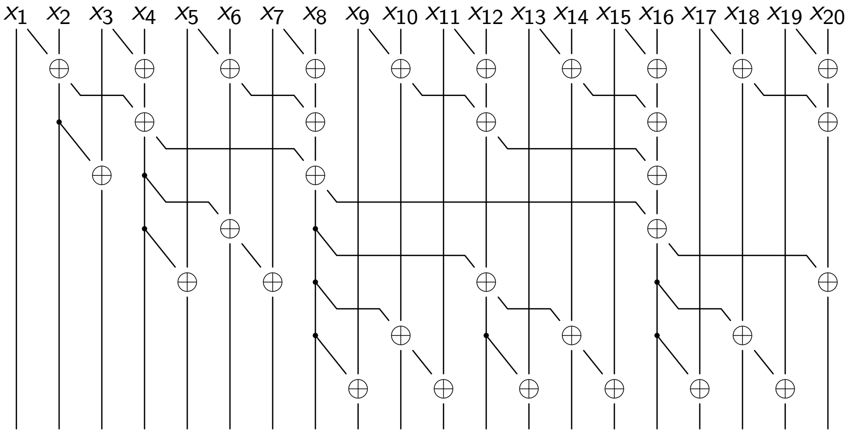

Wanted: reasoning principles, verification techniques, systematic testing approach, ...

Investigating Particular Instances Only

Knuth's 0-1-Principle

If a comparison-swap algorithm sorts correctly on the Booleans, it does so on arbitrary totally ordered value sets.

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Unfortunately not !

A Knuth-like 0-1-2-Principle [V. 2008]

Given: $\texttt{scanl1} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$

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\begin{aligned}
&\texttt{scanl1}\ (\oplus)\ (x : xs) = go\ x\ xs \\
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data Three = Zero | One | Two

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then the same holds for every type τ, $xs :: [\tau]$, and associative $(\oplus) :: \tau \to \tau \to \tau$.

Why 0-1-2? And How?

A question: What can `candidate` $:: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$
do, given an operation $\oplus$ and input list $[x_1, \ldots, x_n]$?

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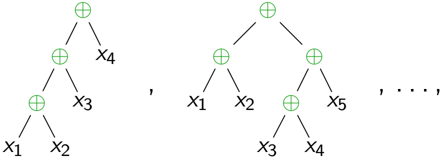
Among these expressions, there are good ones:



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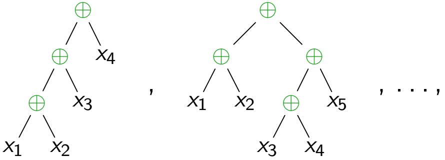

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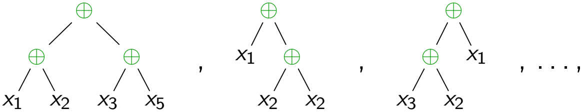

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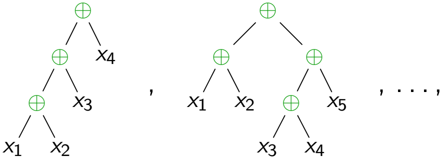

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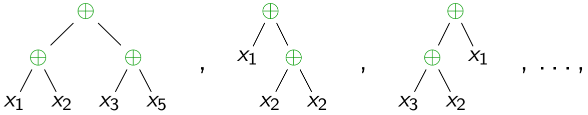
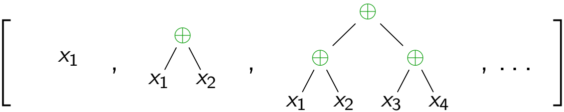


and ones in the wrong position:



That's How!

Let

$\oplus_1$	Zero	One	Two
Zero	Zero	One	Two
One	One	Two	Two
Two	Two	Two	Two

and

$\oplus_2$	Zero	One	Two
Zero	Zero	One	Two
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If `candidate` $(\oplus_1)$ is correct on each list of the form

$$[(\text{Zero},)^* \ \text{One}\ (,\text{Zero})^* \ (,\text{Two})^*]$$

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$$[(\text{Zero},)^* \ \text{One},\text{Two}\ (,\text{Zero})^*]$$

then `candidate` is correct for associative $\oplus$ at arbitrary type.

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then the same holds for every type τ, $xs :: [\tau]$, and associative $(\oplus) :: \tau \to \tau \to \tau$.

Using a Free Theorem (Generator)

Input: `candidate :: (a -> a -> a) -> [a] -> [a]`

Output:
```
forall t1,t2 in TYPES, f :: t1 -> t2.
  forall g :: t1 -> t1 -> t1.
    forall h :: t2 -> t2 -> t2.
      (forall x :: t1. forall y :: t1.
         f (g x y) = h (f x) (f y))
      ==> (forall z :: [t1].
                map f (candidate g z)
                 = candidate h (map f z))
```

Rephrased

For every choice of types τ_1, τ_2 and functions $f :: \tau_1 \to \tau_2$, $(\otimes) :: \tau_1 \to \tau_1 \to \tau_1$, and $(\oplus) :: \tau_2 \to \tau_2 \to \tau_2$:

$$\begin{tikzcd}
\tau_1 \times \tau_1 \arrow[r, "\otimes"] \arrow[d, "f \times f"'] \arrow[dr, phantom, "="] & \tau_1 \arrow[d, "f"] & & {[\tau_1]} \arrow[r, "\texttt{candidate}\ (\otimes)"] \arrow[d, "\texttt{map}\ f"'] \arrow[dr, phantom, "="] & {[\tau_1]} \arrow[d, "\texttt{map}\ f"] \\
\tau_2 \times \tau_2 \arrow[r, "\oplus"'] & \tau_2 \arrow[ur, phantom, "\Rightarrow"] & & {[\tau_2]} \arrow[r, "\texttt{candidate}\ (\oplus)"'] & {[\tau_2]}
\end{tikzcd}$$

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Decomposing the 0-1-2-Principle

Proposition 1: If `candidate` $(\oplus_1)$ is correct on each list of the form $[(\mathrm{Zero},)^* \ \mathrm{One}\ (,\mathrm{Zero})^* \ (,\mathrm{Two})^*]$ and `candidate` $(\oplus_2)$ is correct on each list of the form $[(\mathrm{Zero},)^* \ \mathrm{One}, \mathrm{Two}\ (,\mathrm{Zero})^*]$, then for every $n \geq 0$,
`candidate` $(+\!\!+)$ $[[k] \mid k \leftarrow [0..n]] \ = \ [[0..k] \mid k \leftarrow [0..n]]$ $(*)$.

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\begin{tikzcd}
\tau_1 \times \tau_1 \arrow[r, "\otimes"] \arrow[d, "f \times f"'] \arrow[dr, phantom, "="] & \tau_1 \arrow[d, "f"] \\
\tau_2 \times \tau_2 \arrow[r, "\oplus"'] & \tau_2
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
{[\tau_1]} \arrow[r, "\texttt{candidate}\ (\otimes)"] \arrow[d, "\texttt{map}\ f"'] \arrow[dr, phantom, "="] & {[\tau_1]} \arrow[d, "\texttt{map}\ f"] \\
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{[\mathrm{Int}] \times [\mathrm{Int}]} \arrow[r, "+\!\!+"] \arrow[d, "f \times f"'] \arrow[dr, phantom, "="] & {[\mathrm{Int}]} \arrow[d, "f"] \\
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{[[k] \mid k \leftarrow [0..n]]} \arrow[d, phantom, "::"] & \\
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- For parallel prefix computation, formalisation available in Isabelle/HOL [Böhme 2007].

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- For which other algorithm classes can one play the same trick?

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Excerpt from Formal Proof — Where Associativity is Used

Let $xs :: [\tau_2]$ with length $(n + 1)$.

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