Discovering Counterexamples (and Proof Ingredients) for Knuth-like 0-1-…-k-Principles

Moritz Fürneisen and Janis Voigtländer

University of Bonn

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Knuth's 0-1-Principle [Knuth 1973]

Informally: If a comparison-swap algorithm sorts Booleans correctly, it sorts integers correctly as well.

Formally: Let (in Haskell, say):

$$\texttt{sort} :: ((\alpha,\alpha) \to (\alpha,\alpha)) \to [\alpha] \to [\alpha]$$

$$f :: (\mathrm{Int}, \mathrm{Int}) \to (\mathrm{Int}, \mathrm{Int})$$
$$f\ (x,y) = \mathit{if}\ x > y\ \mathit{then}\ (y,x)\ \mathit{else}\ (x,y)$$

$$g :: (\mathrm{Bool}, \mathrm{Bool}) \to (\mathrm{Bool}, \mathrm{Bool})$$
$$g\ (x,y) = (x\ \&\&\ y, x \mid\mid y)$$

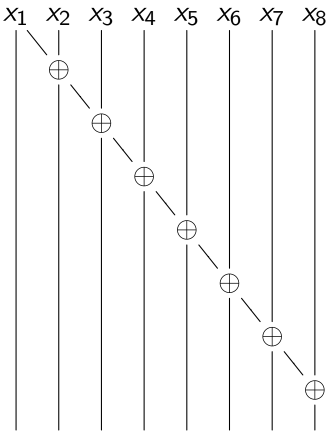
If for every xs :: [Bool], `sort` g xs gives the correct result, then for every xs :: [Int], `sort` f xs gives the correct result.

Parallel Prefix Computation

Given: inputs $x_1, \ldots, x_n$ and an associative operation $\oplus$

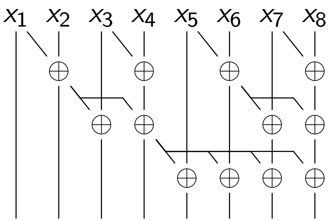
Task: compute the values $x_1, x_1 \oplus x_2, \ldots, x_1 \oplus x_2 \oplus \cdots \oplus x_n$

Solution:

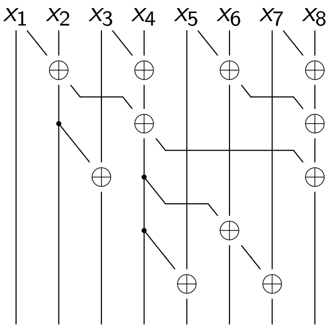


Parallel Prefix Computation

Alternative:



Or:



Or: . . .

Expressing Parallel Prefix Algorithms in Haskell

Functions of type:

$$\texttt{scanl1} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$$

For example, à la [Sklansky 1960]:

$$
\begin{array}{llll}
\texttt{sklansky} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha] & & & \\
\texttt{sklansky}\ (\oplus)\ [x] = [x] & & & \\
\texttt{sklansky}\ (\oplus)\ \mathit{xs}\ = \mathit{us} \mathbin{+\!\!+} \mathit{vs} & & & \\
\quad \textbf{where}\ t & & = & ((\texttt{length}\ \mathit{xs}) + 1)\ `\texttt{div}`\ 2 \\
\quad & (\mathit{ys}, \mathit{zs}) & = & \texttt{splitAt}\ t\ \mathit{xs} \\
\quad & \mathit{us} & = & \texttt{sklansky}\ (\oplus)\ \mathit{ys} \\
\quad & \mathit{vs} & = & [(\texttt{last}\ \mathit{us}) \oplus v \mid v \leftarrow \texttt{sklansky}\ (\oplus)\ \mathit{zs}]
\end{array}
$$

Investigating Particular Instances Only

Knuth's 0-1-Principle

If a comparison-swap algorithm sorts correctly on the Booleans, it does so on arbitrary totally ordered value sets.

A Knuth-like 0-1-Principle ?

If a parallel prefix algorithm is correct (for associative operations) on the Booleans, it is so on arbitrary value sets.

Unfortunately not !

A Knuth-like 0-1-2-Principle [V. 2008]

Given: $\texttt{scanl1} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$

$\texttt{scanl1}\ (\oplus)\ (x : xs) = go\ x\ xs$

where $go\ x\ [\,] \qquad = [x]$

$go\ x\ (y : ys) = x : (go\ (x \oplus y)\ ys)$

$\texttt{candidate} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$

data Three = Zero | One | Two

Theorem: If for every xs :: [Three] and associative $(\oplus)$ :: Three $\to$ Three $\to$ Three,

$$\texttt{candidate}\ (\oplus)\ xs \quad = \quad \texttt{scanl1}\ (\oplus)\ xs\,,$$

then the same holds for every type τ, $xs :: [\tau]$, and associative $(\oplus) :: \tau \to \tau \to \tau$.

Why 0-1-2? And How?

- To get going, proof uses parametricity [Reynolds 1983], deriving a "free theorem" [Wadler 1989].
- Proof crucially involves two specific associative functions:

$\oplus_1$	Zero	One	Two
Zero	Zero	One	Two
One	One	Two	Two
Two	Two	Two	Two

and

$\oplus_2$	Zero	One	Two
Zero	Zero	One	Two
One	One	One	Two
Two	Two	One	Two

  and their behavior on lists $[(\text{Zero},)^* \text{ One } (,\text{Zero})^* \ (,\text{Two})^*]$ and $[(\text{Zero},)^* \text{ One}, \text{Two } (,\text{Zero})^*]$, respectively.

- Formalisation available in Isabelle/HOL [Böhme 2007].

But:

- Does that really explain the why and how?
- What to do to get similar results for other algorithm classes?

Plan of the Talk

- Start telling the story of how "0-1-2", $\oplus_1$, $\oplus_2$, ... were found (back in 2007, never recorded, but interesting I think).

- Challenge you to suggest other approaches to discover the required counterexamples and proof ingredients?

- Invite complaints about where the presented (deliberately naive, exploratory) approach is too ad-hoc, or unacceptably pulls a rabbit out of a hat.

- Provoke investigation/proposals of other algorithm classes on which one could try to play the same or similar trick(s)?

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Unfortunately not !

Let's:

- try to find out why not,
- and why 0-1-2 makes more sense to attempt proving,
- and let's try to do all that without pulling too many rabbits.

What a Knuth-like 0-1-Principle Would Be

Given: $\texttt{scanl1} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$

$\texttt{scanl1}\ (\oplus)\ (x : xs) = go\ x\ xs$

$\quad$ **where** $go\ x\ [\,] \qquad\quad = [x]$

$\qquad\qquad go\ x\ (y : ys) = x : (go\ (x \oplus y)\ ys)$

$\texttt{candidate} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$

Theorem?: If for every associative $(\oplus) :: \mathrm{Bool} \to \mathrm{Bool} \to \mathrm{Bool}$ and $xs :: [\mathrm{Bool}]$,

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then the same holds for every type τ, associative $(\oplus) :: \tau \to \tau \to \tau$, and $xs :: [\tau]$.

Let's try to find a counterexample by property-based testing.

Getting Started

Somewhat naive expression of our intent:

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\texttt{quickCheck}\ \$\ \lambda(\texttt{candidate} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]) \to \\
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\qquad \lambda \mathit{xs} :: [\mathsf{Bool}] \to \texttt{candidate}\ (\oplus)\ \mathit{xs} == \texttt{scanl1}\ (\oplus)\ \mathit{xs}) \\
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- generating (polymorphic) functions?
- randomness vs. exhaustiveness?
- a complex property as precondition of "==>"?
- generating/picking the type τ?

Transforming Polymorphism Away

Let's try to get a grip on `candidate` $:: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$.

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So, to any `candidate` as above corresponds a function of type $\text{Nat} \to [\text{Expr}]$, where

$$\textbf{data}\ \text{Expr} = \text{Var Nat} \mid \text{Op Expr Expr}$$

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But dependent on the input list length n !

So, to any `candidate` as above corresponds a function of type Nat $\to$ [Expr], where

data Expr = Var Nat | Op Expr Expr

We can prove this (almost) isomorphism using parametricity/"free theorems" [Reynolds 1983, Wadler 1989], and can actually program appropriate conversions.

Where Are We Now?

To test:

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That idea is basically fine here, but we actually need some more control because of our need to check the complex precondition.

Making Partiality Explicit (and "Fixing the Default")

We go from $\mathsf{Nat} \to [\mathsf{Expr}]$ to $\mathsf{Nat} \to \mathsf{Maybe}\,[\mathsf{Expr}]$,
with the interpretation that

$$\texttt{representation}\ n = \mathsf{Nothing}$$

means to take on the `scanl1`-behavior for lists of length n.

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For example,

$$\begin{array}{llll}
\texttt{representation} :: \text{Nat} \to \text{Maybe}\ [\text{Expr}] & & & \\
\texttt{representation}\ n & \mid n == 1 & = & \text{Just}\ [\text{Op}\ (\text{Var}\ 0)\ (\text{Var}\ 0)] \\
 & \mid n == 3 & = & \text{Just}\ [\text{Var}\ 0 \\
 & & & \quad ,\text{Op}\ (\text{Var}\ 1)\ (\text{Var}\ 2)] \\
 & \mid \mathit{otherwise} & = & \text{Nothing}
\end{array}$$

corresponds to

$$\begin{array}{lll}
\texttt{candidate} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha] & & \\
\texttt{candidate}\ (\oplus)\ [x] & = & [x \oplus x] \\
\texttt{candidate}\ (\oplus)\ [x, y, z] & = & [x, y \oplus z] \\
\texttt{candidate}\ (\oplus)\ \mathit{xs} & = & \texttt{scanl1}\ (\oplus)\ \mathit{xs}
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type Sparse = [(Nat, [Expr])]

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type Sparse = [(Nat, [Expr])]
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Previous example then corresponds to (among others):

```
sparse :: Sparse
sparse = [(1, [Op (Var 0) (Var 0)])
         , (3, [Var 0, Op (Var 1) (Var 2)])]
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Actually a technical challenge now: programmatic conversion between [(Nat, [Expr])] and Nat → Maybe [Expr] ?

Excurse: from $\mathsf{Nat} \to \mathsf{Maybe}\,[\mathsf{Expr}]$ to $[(\mathsf{Nat}, [\mathsf{Expr}])]$

Problem:

- A naive “sparsification”, iterating through all natural numbers and keeping those for which the result is non-Nothing, does not work well.

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Solution:

- work with lazy natural numbers: **data** Nat $=$ *Z* | *S* Nat

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Solution:

- work with lazy natural numbers: **data** Nat $= Z \mid S$ Nat
- use an appropriately lazy function for lookup in [(Nat, [Expr])]

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- work with lazy natural numbers: **data** Nat = *Z* | *S* Nat
- use an appropriately lazy function for lookup in [(Nat, [Expr])]
- convert with care

Excurse: from $\text{Nat} \rightarrow \text{Maybe}\,[\text{Expr}]$ to $[(\text{Nat}, [\text{Expr}])]$

Problem:

- A naive “sparsification”, iterating through all natural numbers and keeping those for which the result is non-Nothing, does not work well.
- For example, `sparsify` (`const` Nothing) would simply give $\perp$ :: Sparse, from which we could not learn anything.

Solution:

- work with lazy natural numbers: **data** Nat = $Z \mid S$ Nat
- use an appropriately lazy function for lookup in $[(\text{Nat}, [\text{Expr}])]$
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- (for convenience, actually store only the “gap lengths”)

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Example:

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representation :: Nat → Maybe [Expr]
representation n | n == 1    = Just [Op (Var 0) (Var 0)]
                 | n == 3    = Just [Var 0, ...]
                 | otherwise = Nothing
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now turns into:

```
sparse :: Sparse
sparse = [(S Z, [Op (Var 0) (Var 0)])
         ,(S Z, [Var 0, ...])
         ,(S (S (S ...)), ⊥)] ++ ⊥
```

Wrapping up the Transformation

We now have:

$$
\begin{array}{l}
\texttt{quickCheck}\ \$\ \lambda(\texttt{sparse} :: \mathsf{Sparse}) \to \\
\quad (\texttt{forAll associative}\ \$\ \lambda((\oplus) :: \mathsf{Bool} \to \mathsf{Bool} \to \mathsf{Bool}) \to \\
\qquad \lambda \mathit{xs} :: [\mathsf{Bool}] \to \ldots) \\
\quad {=}{=}{>} \\
\quad \forall \tau.\ (\texttt{forAll associative}\ \$\ \lambda((\oplus) :: \tau \to \tau \to \tau) \to \\
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Now we can:

- implement the complex precondition check
  (by handcoding or using SmallCheck machinery)
- generalize to other finite types
- experiment with different choices for τ
- experiment with different generators for Sparse-candidates
- experiment with QuickCheck vs. SmallCheck

Some Counterexamples Thus Found

$$\begin{aligned}
&\texttt{candidate} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha] \\
&\texttt{candidate}\ (\oplus)\ [x_0] &&= [x_0] \\
&\texttt{candidate}\ (\oplus)\ [x_0, x_1] &&= [x_0, (((x_0 \oplus x_1) \oplus x_0) \oplus x_0) \oplus (x_1 \oplus x_1)] \\
&\texttt{candidate}\ (\oplus)\ xs &&= \texttt{scanl1}\ (\oplus)\ xs
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Recall: Plan of the Talk

- Start telling the story of how "0-1-2", $\oplus_1$, $\oplus_2$, ... were found (back in 2007, never recorded, but interesting I think).
- Challenge you to suggest other approaches to discover the required counterexamples and proof ingredients?
- Invite complaints about where the presented (deliberately naive, exploratory) approach is too ad-hoc, or unacceptably pulls a rabbit out of a hat.
- Provoke investigation/proposals of other algorithm classes on which one could try to play the same or similar trick(s)?

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