Knuth's 0-1-Principle and Beyond

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University of Bonn

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The Sorting Problem

Task: Given a list and an order on the type of elements of this list, produce a sorted list (with same content)!

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Many Solutions:

- Quicksort
- Insertion Sort
- Merge Sort
- Bubble Sort
- ...

Quicksort

1. Choose an element `x` from the input list.
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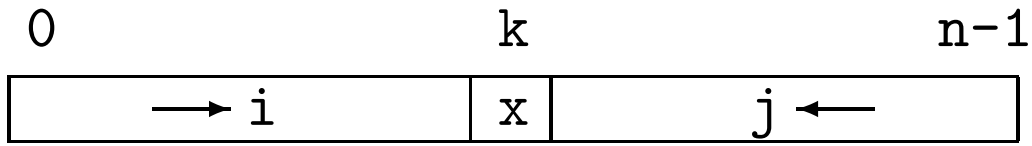

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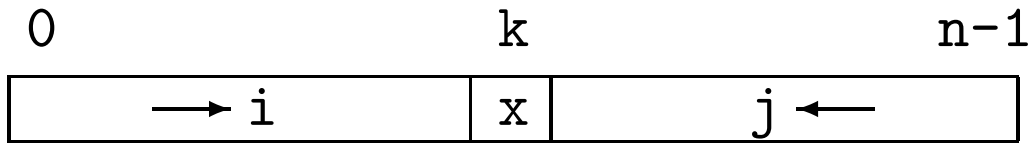

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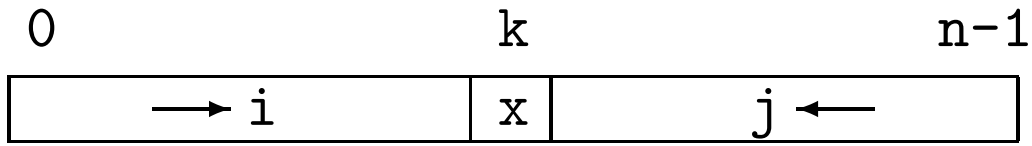

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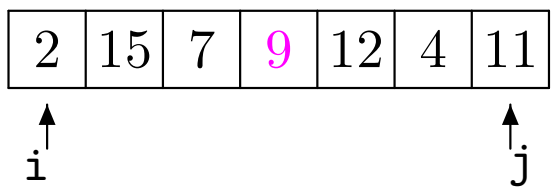

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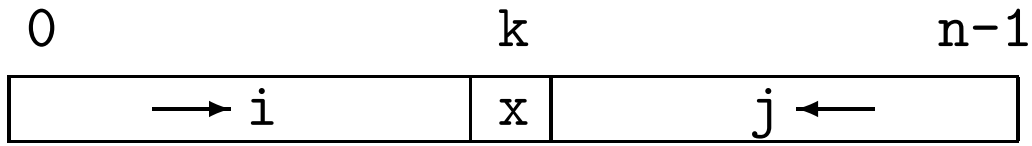

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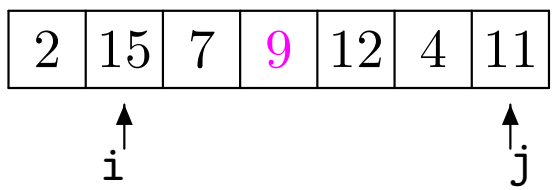

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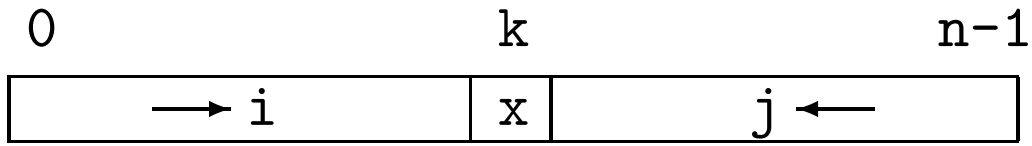

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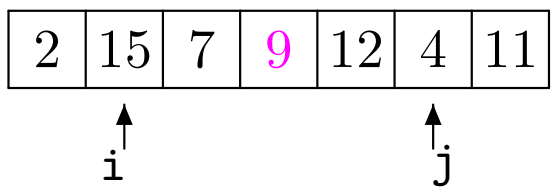

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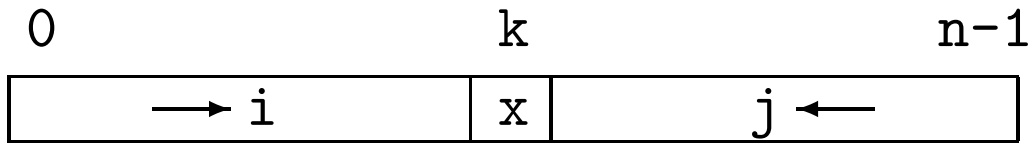

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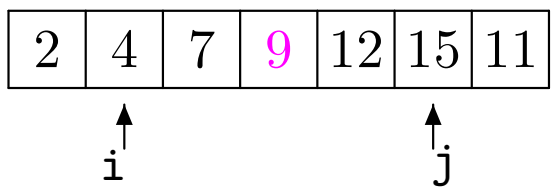

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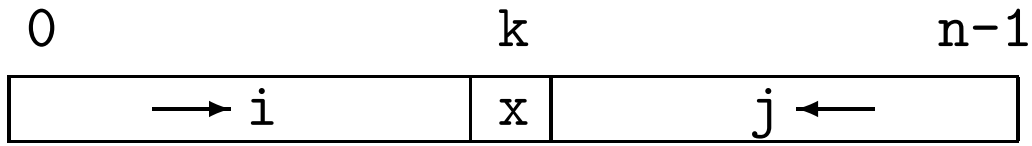

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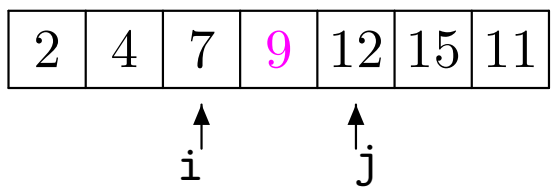

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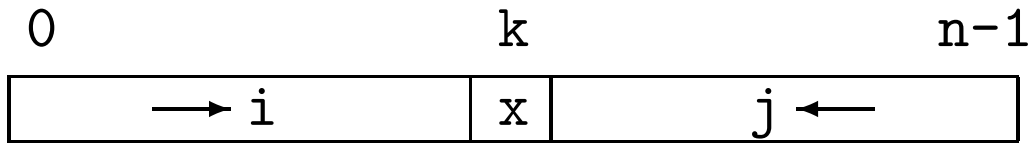

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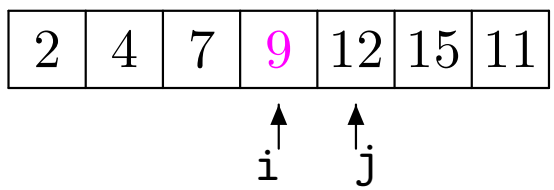

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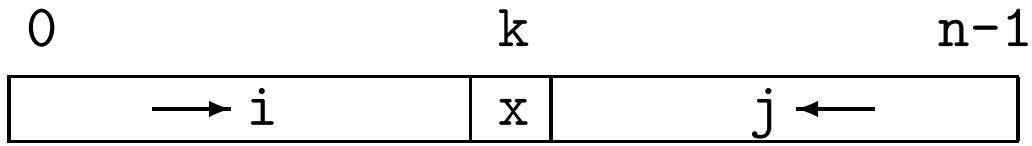

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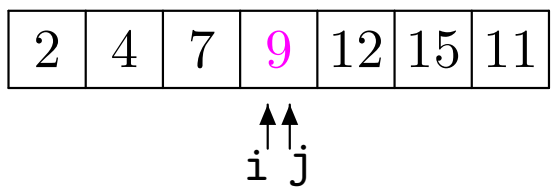

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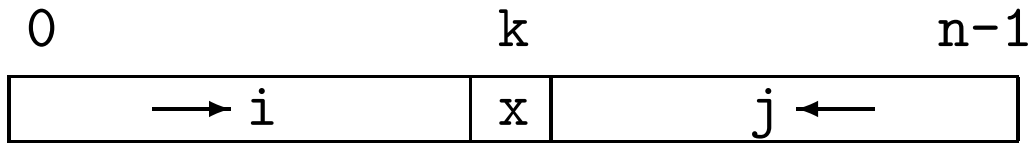

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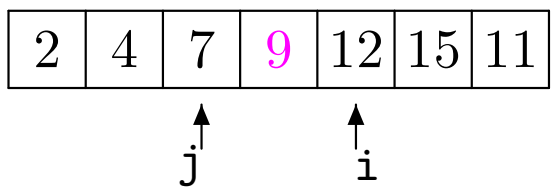

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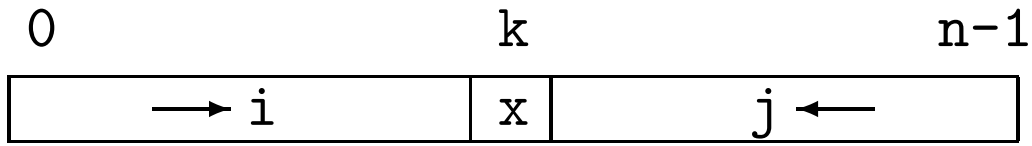

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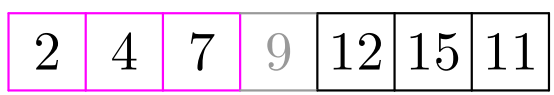

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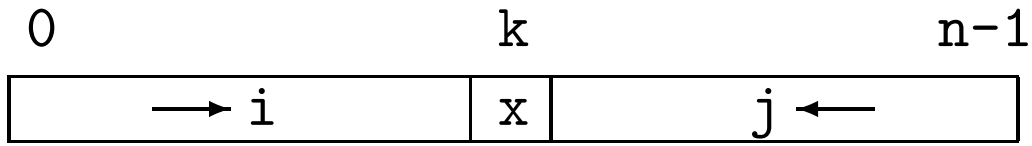

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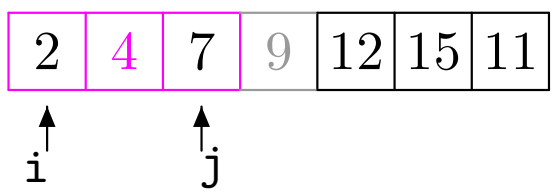

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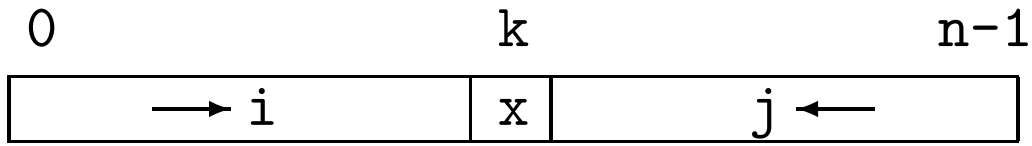

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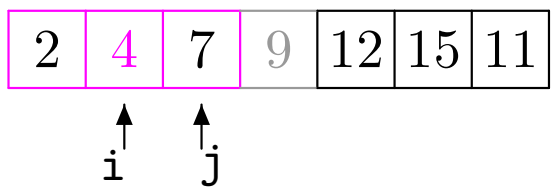

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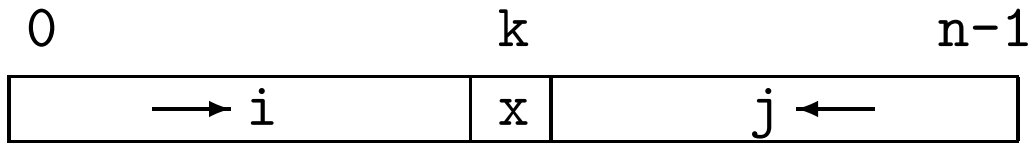

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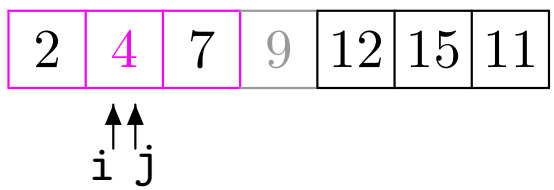

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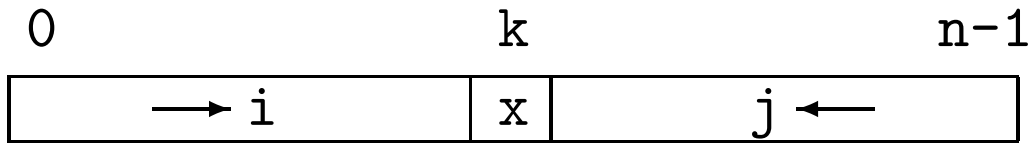

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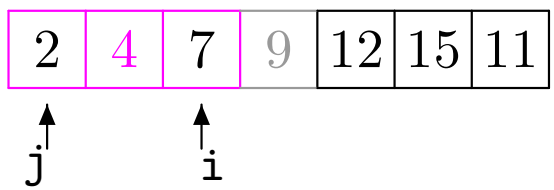

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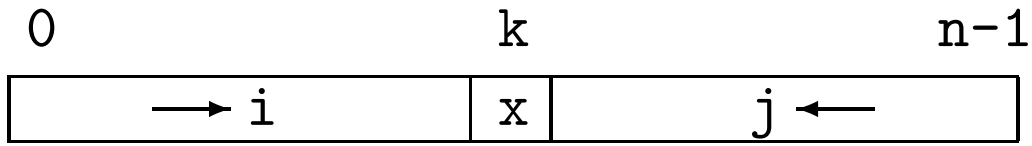

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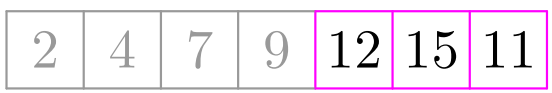

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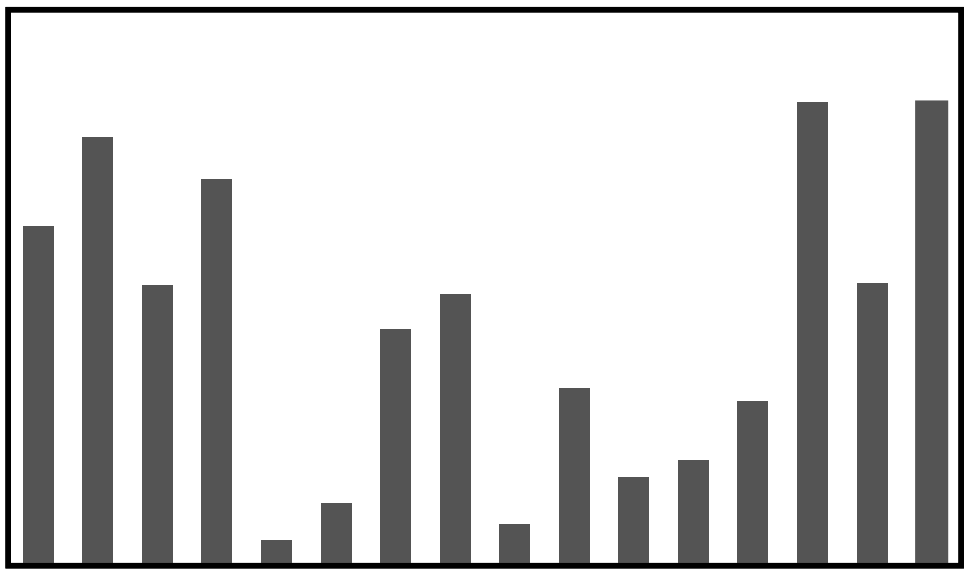

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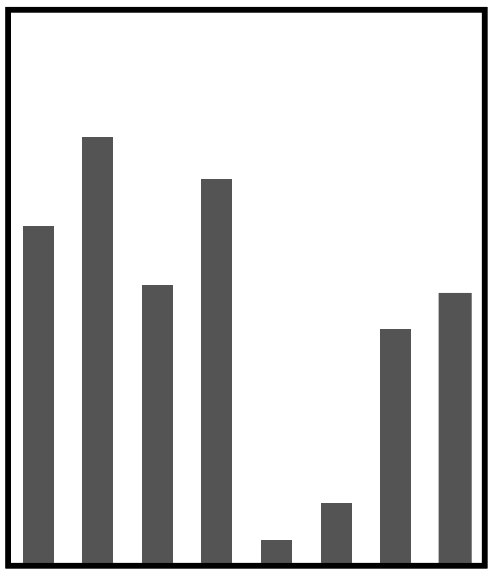

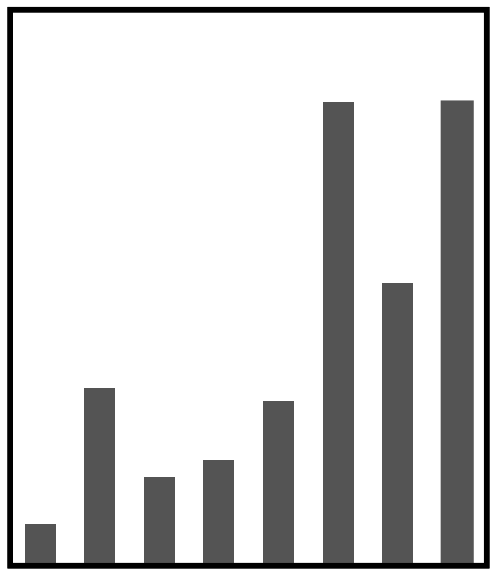

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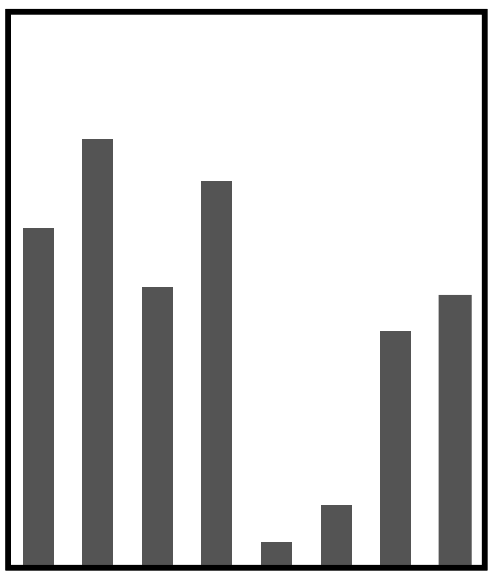

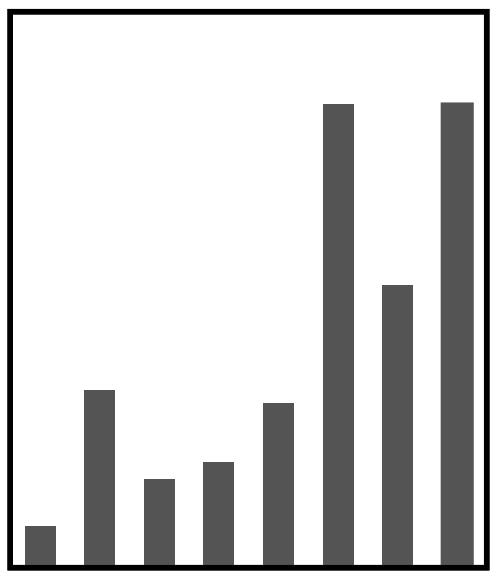

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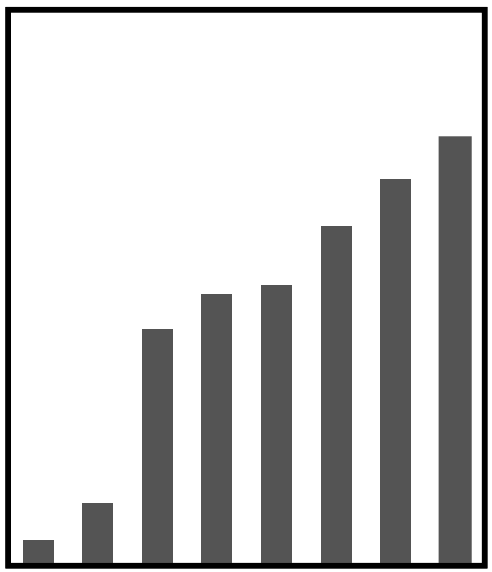

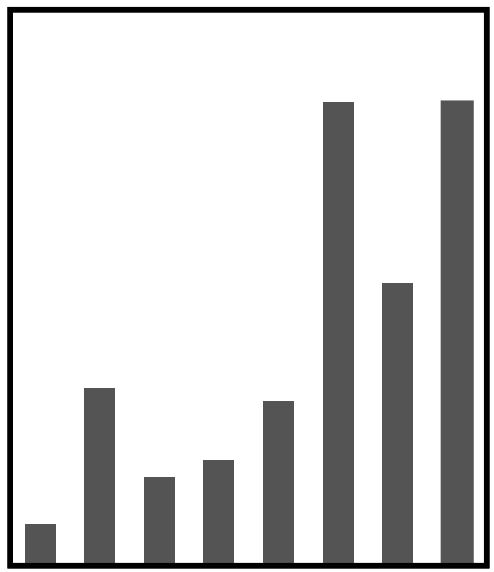

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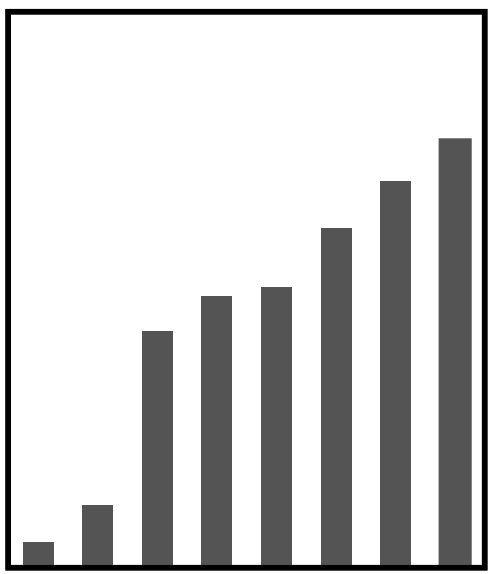

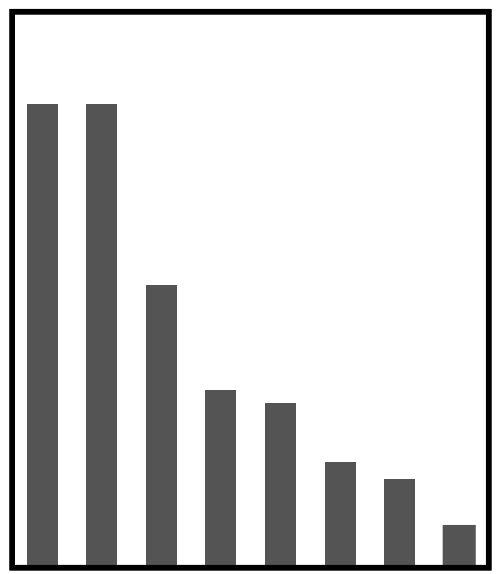

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3. Merge the sorted sublists as follows:
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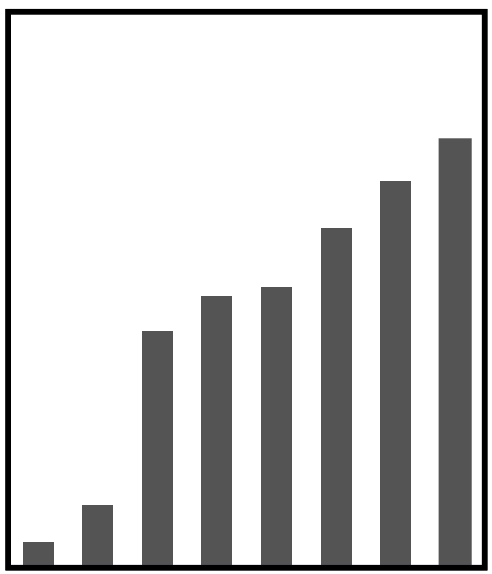

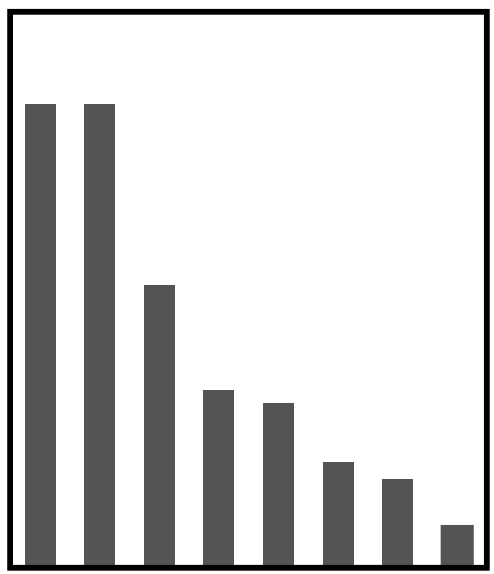

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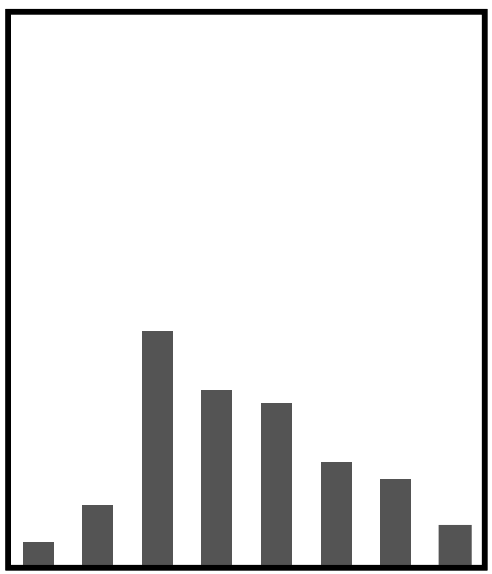

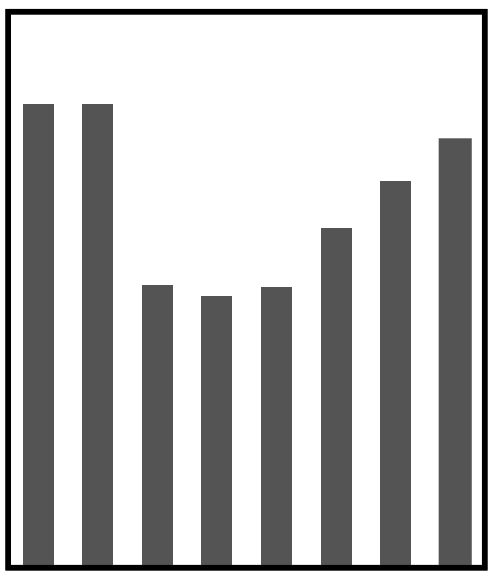

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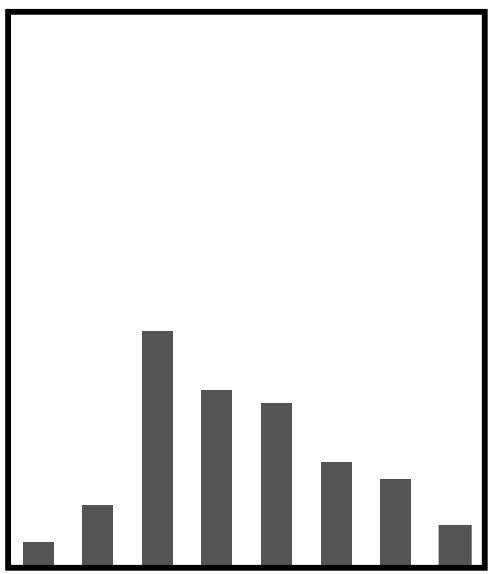

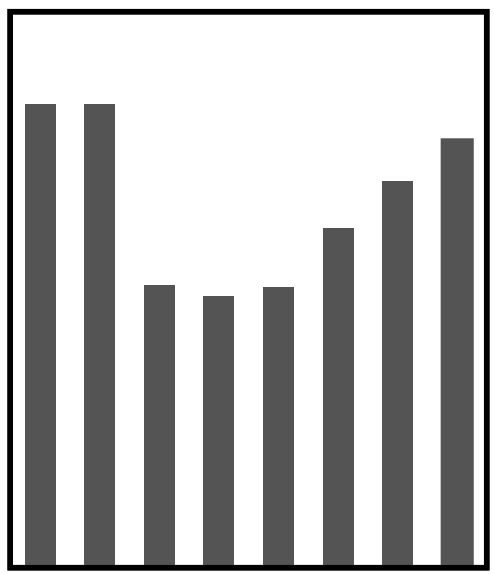

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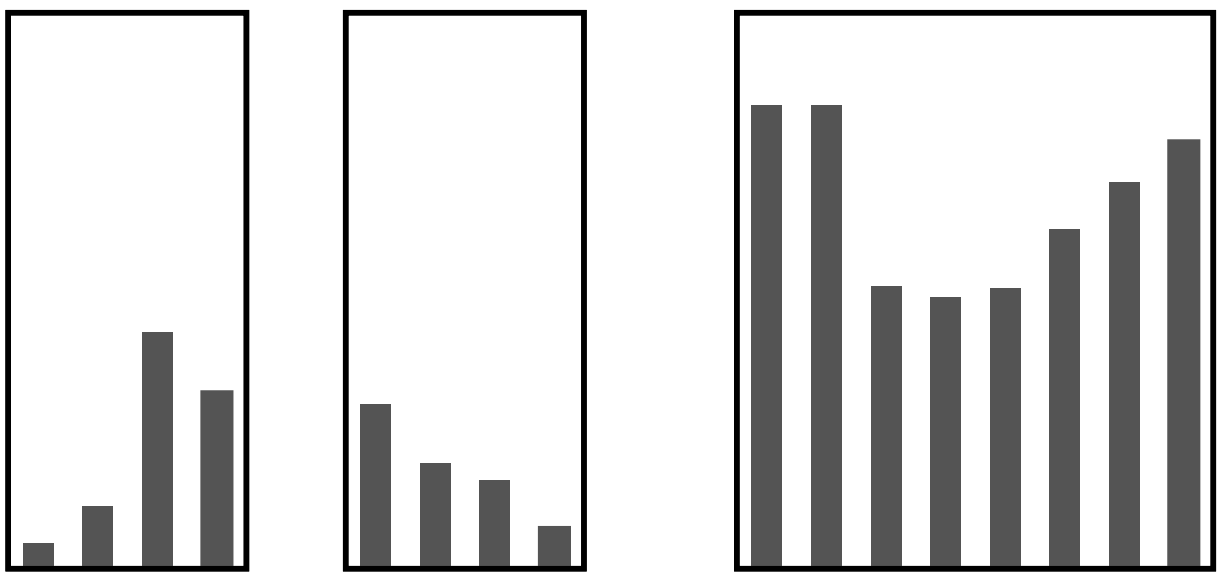

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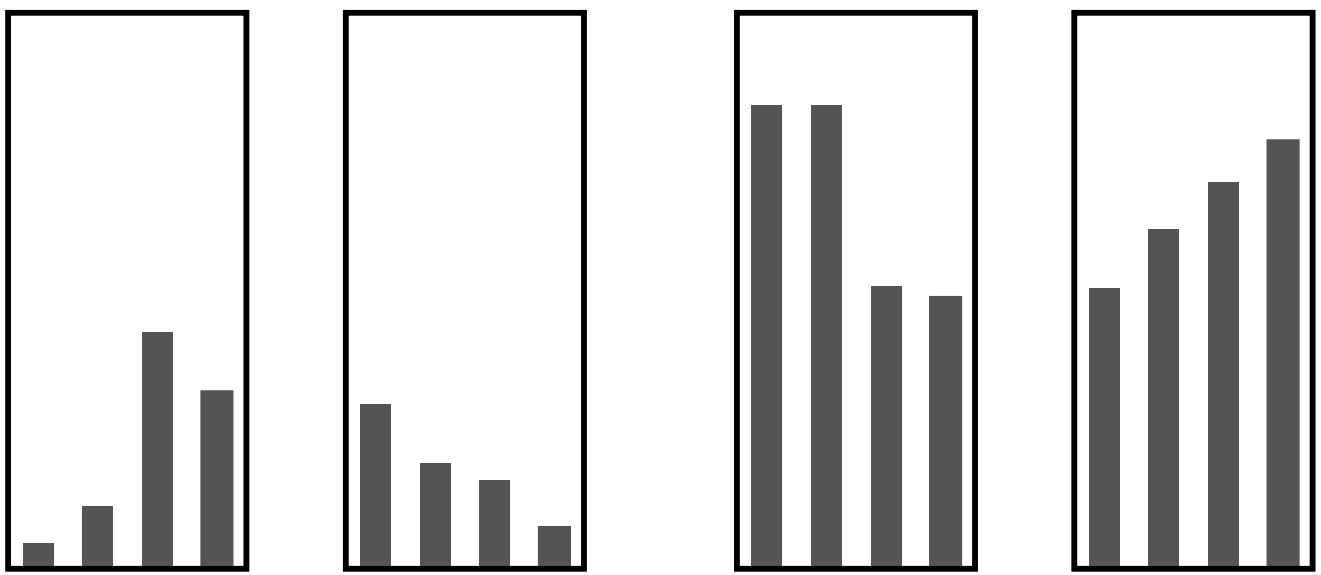

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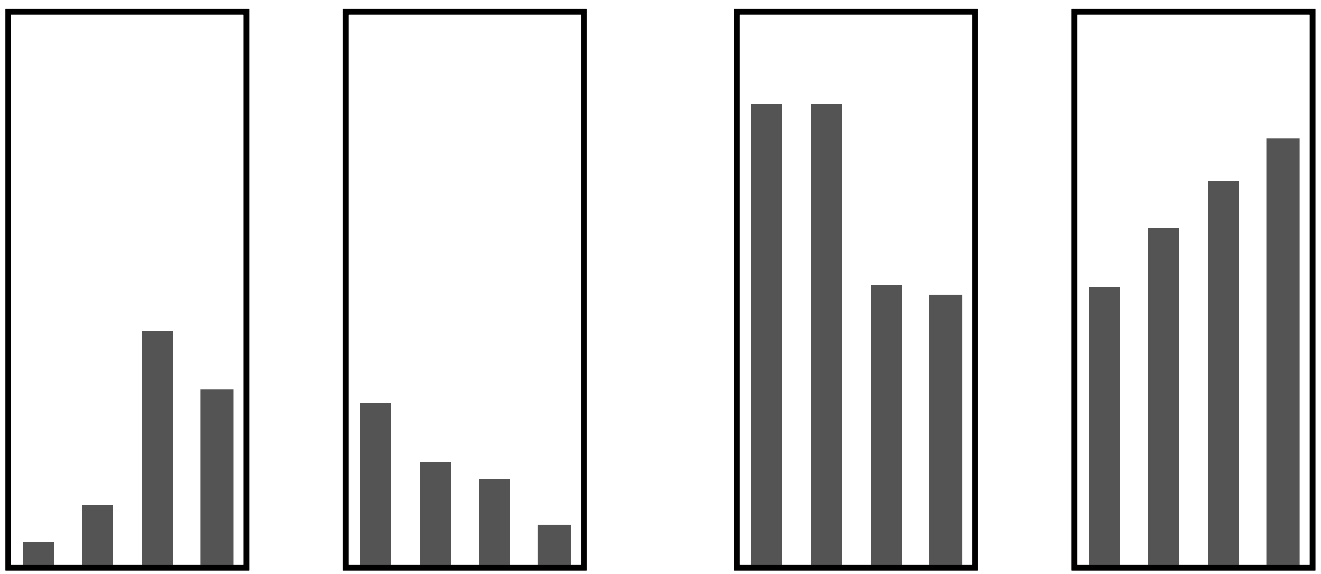

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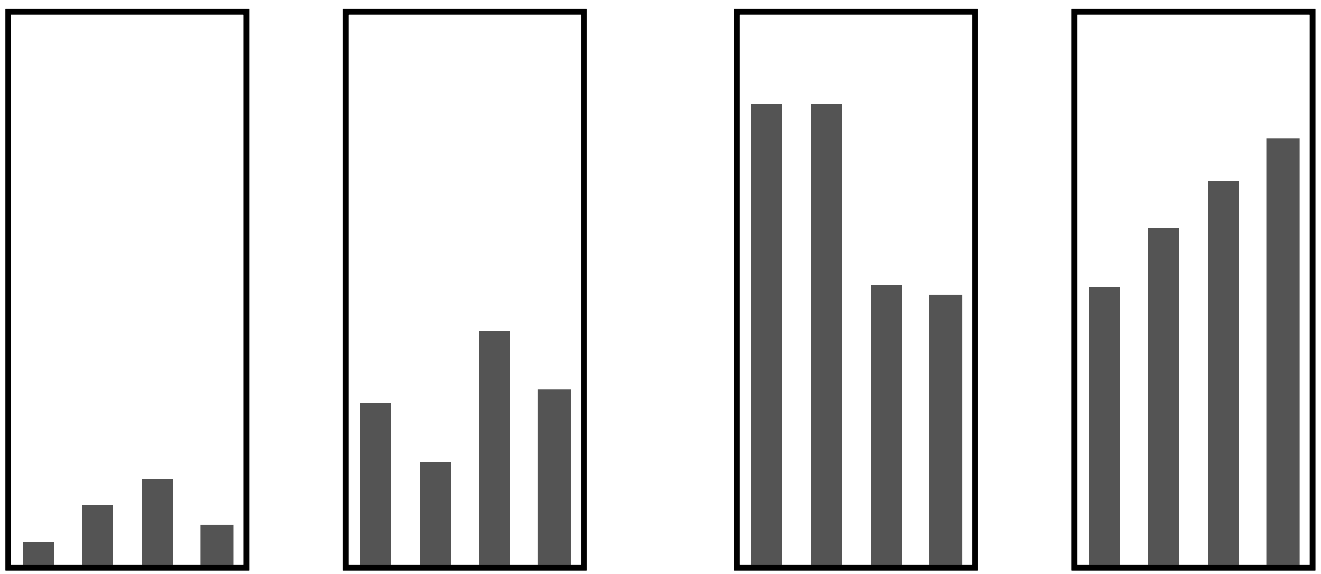

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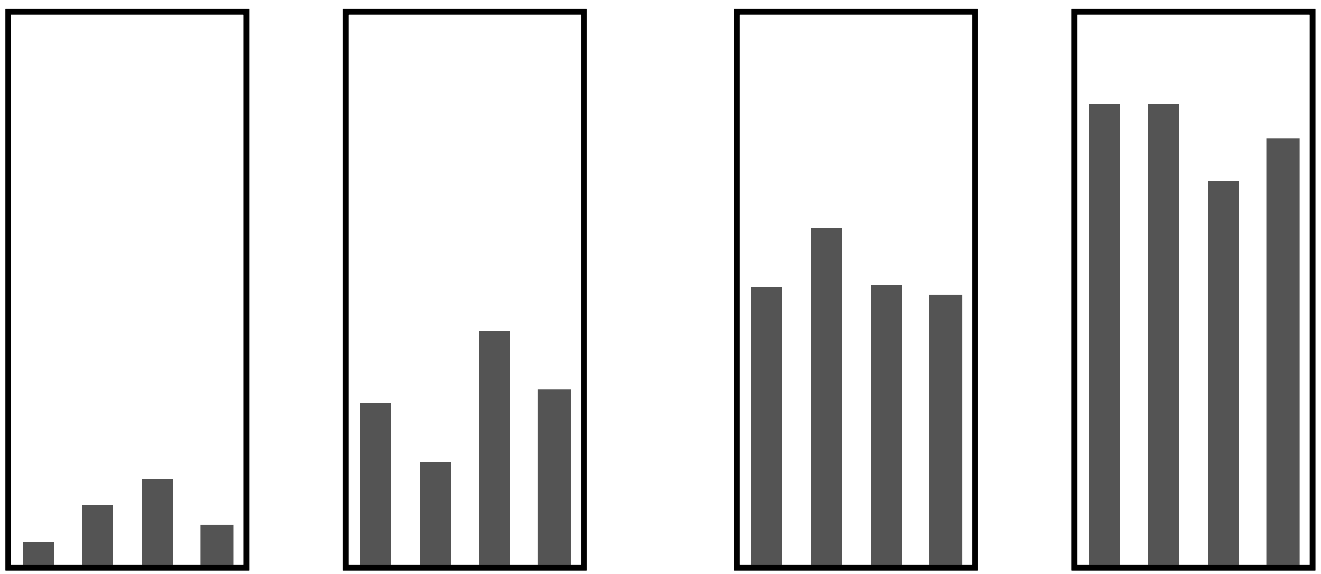

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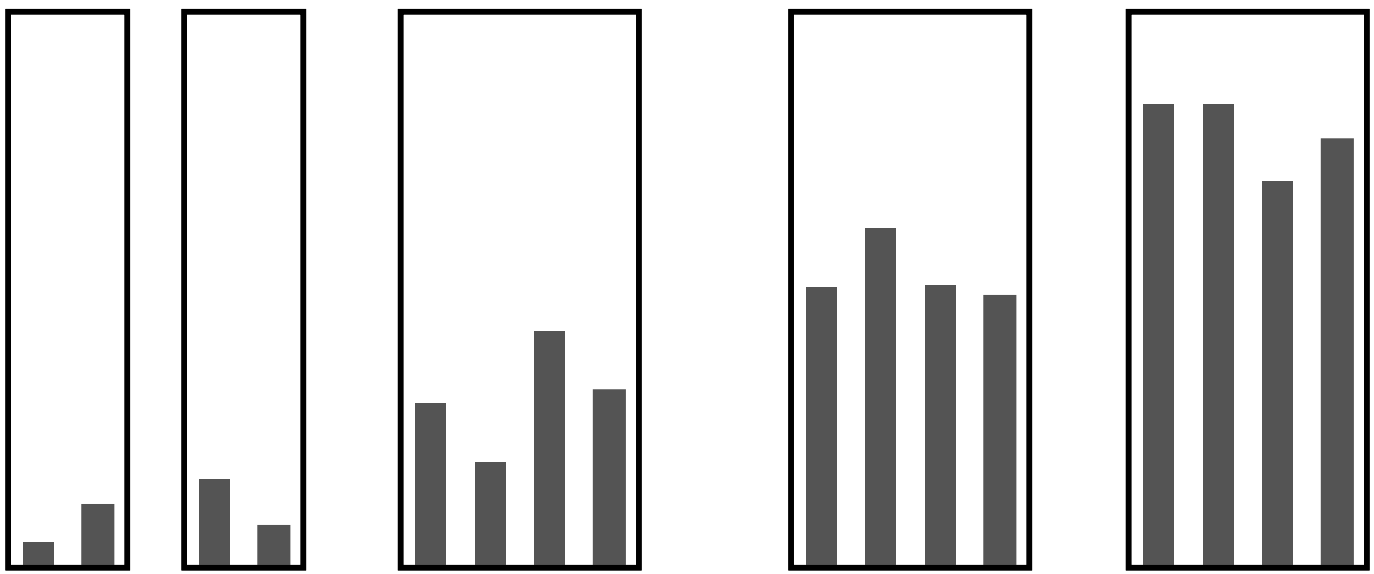

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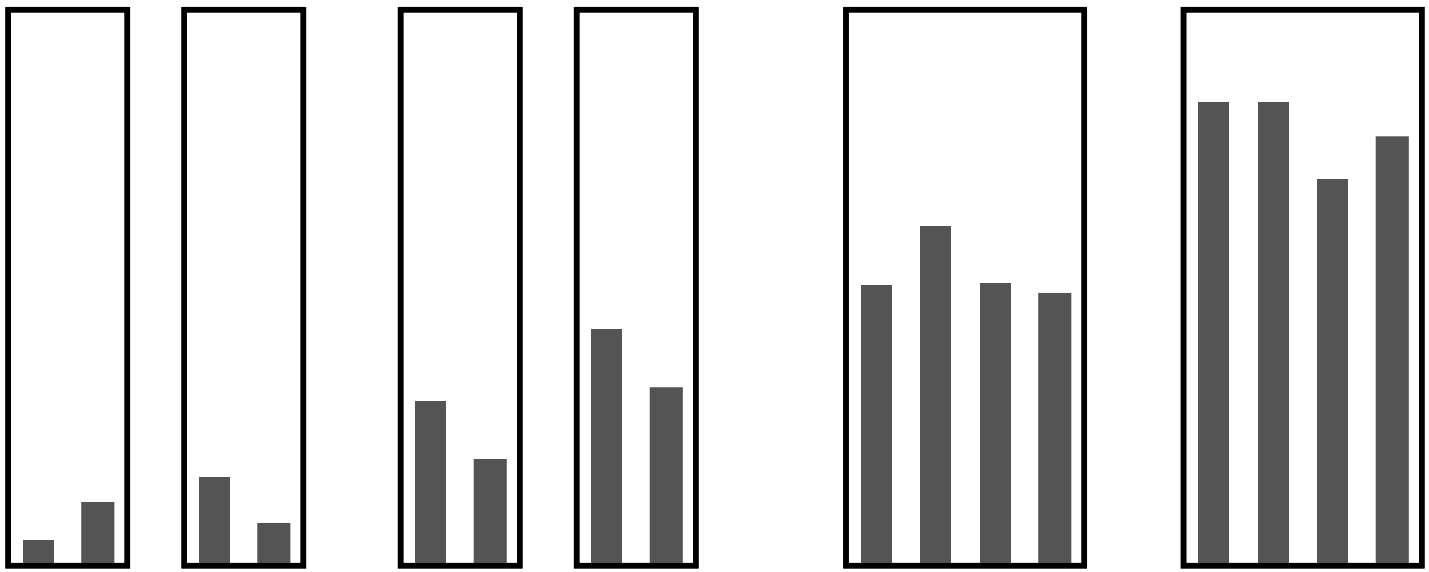

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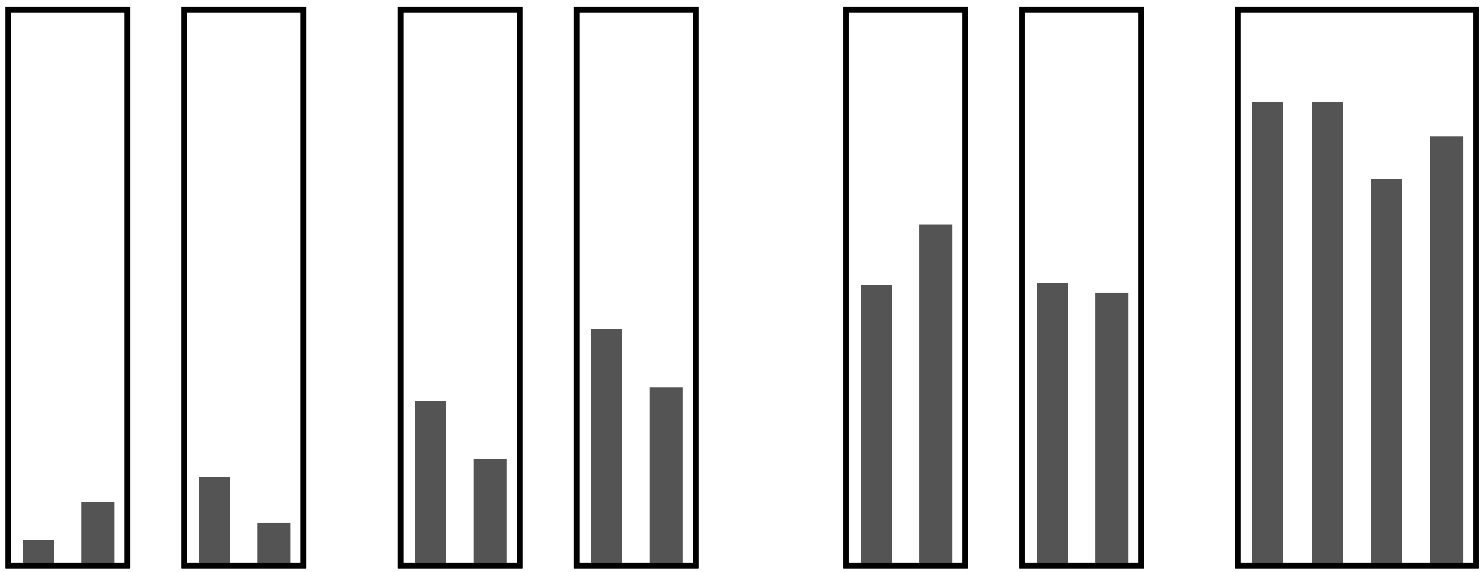

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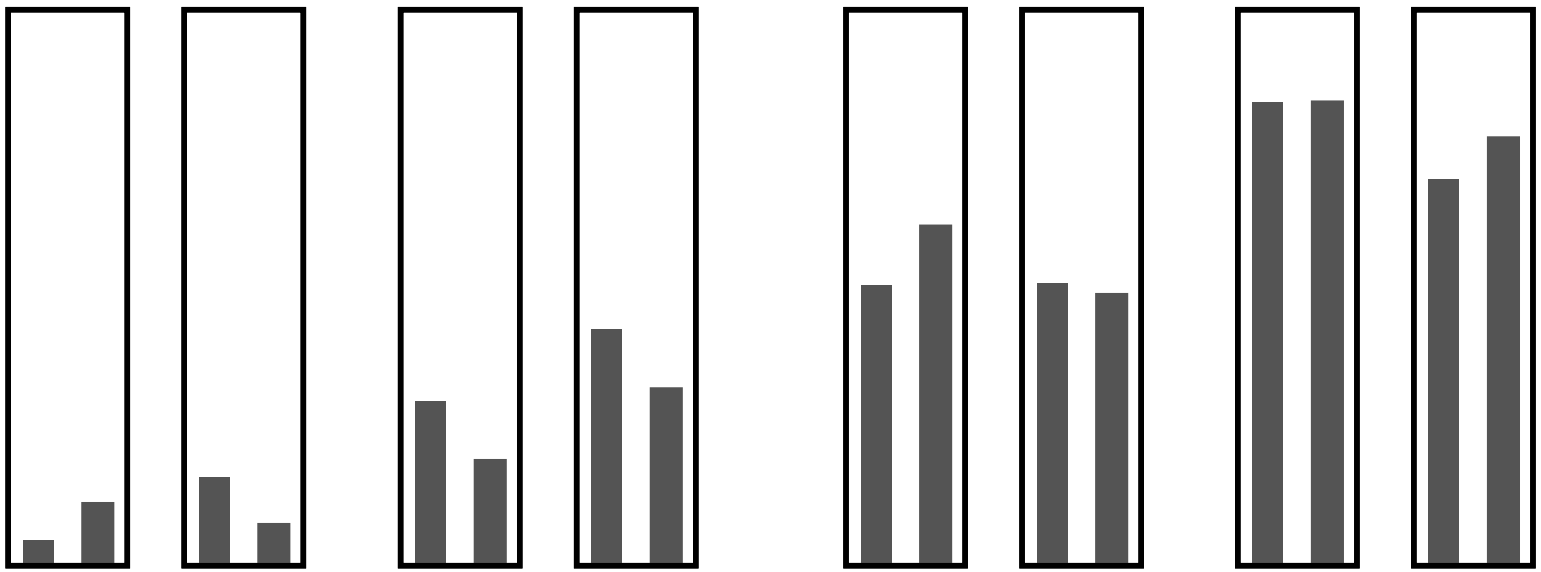

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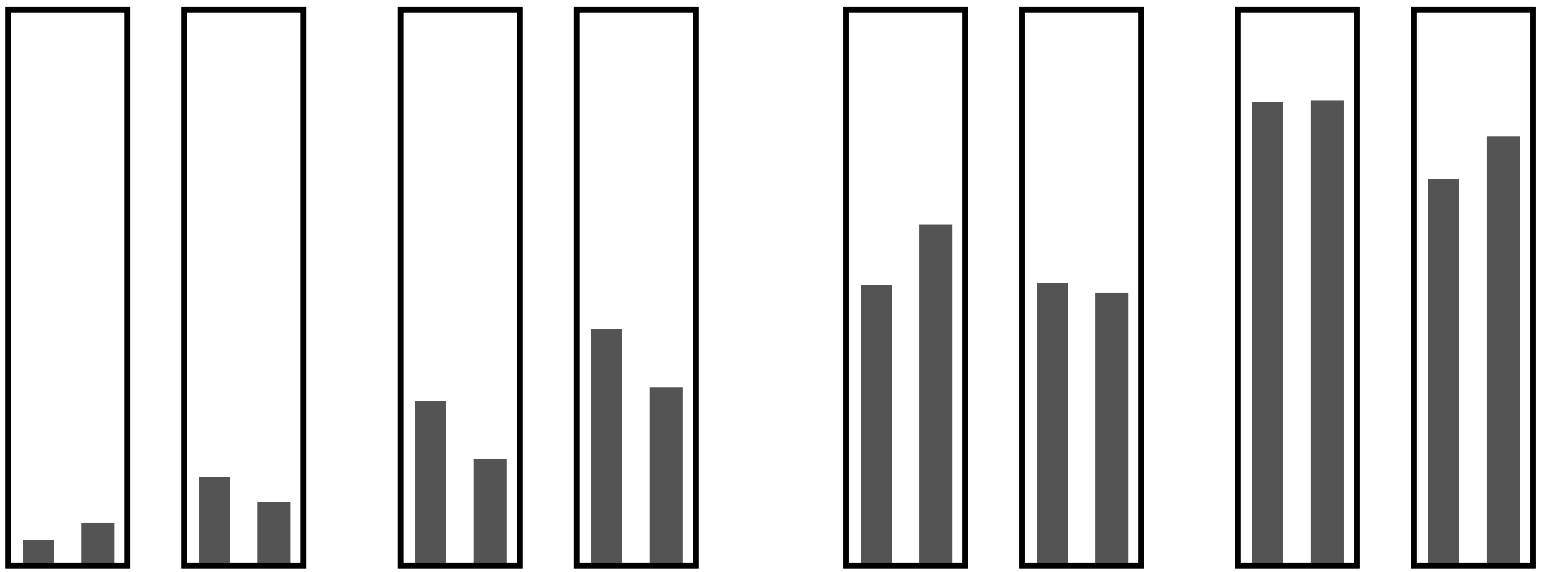

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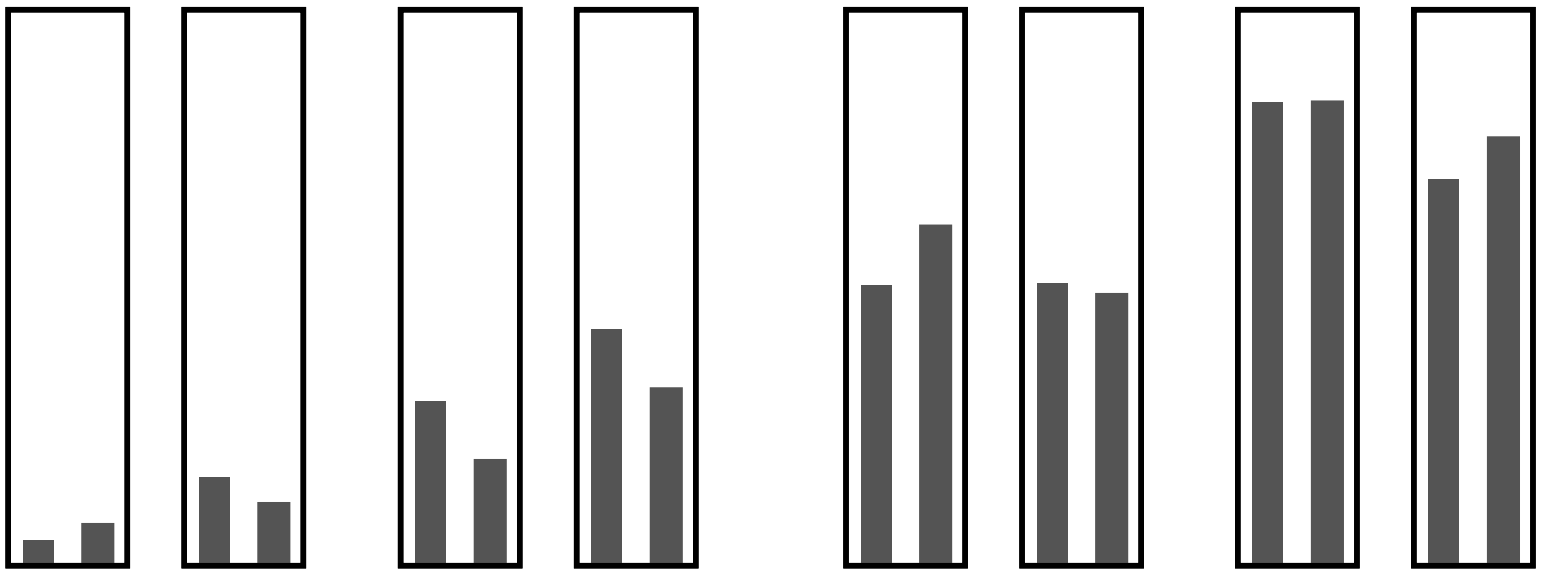

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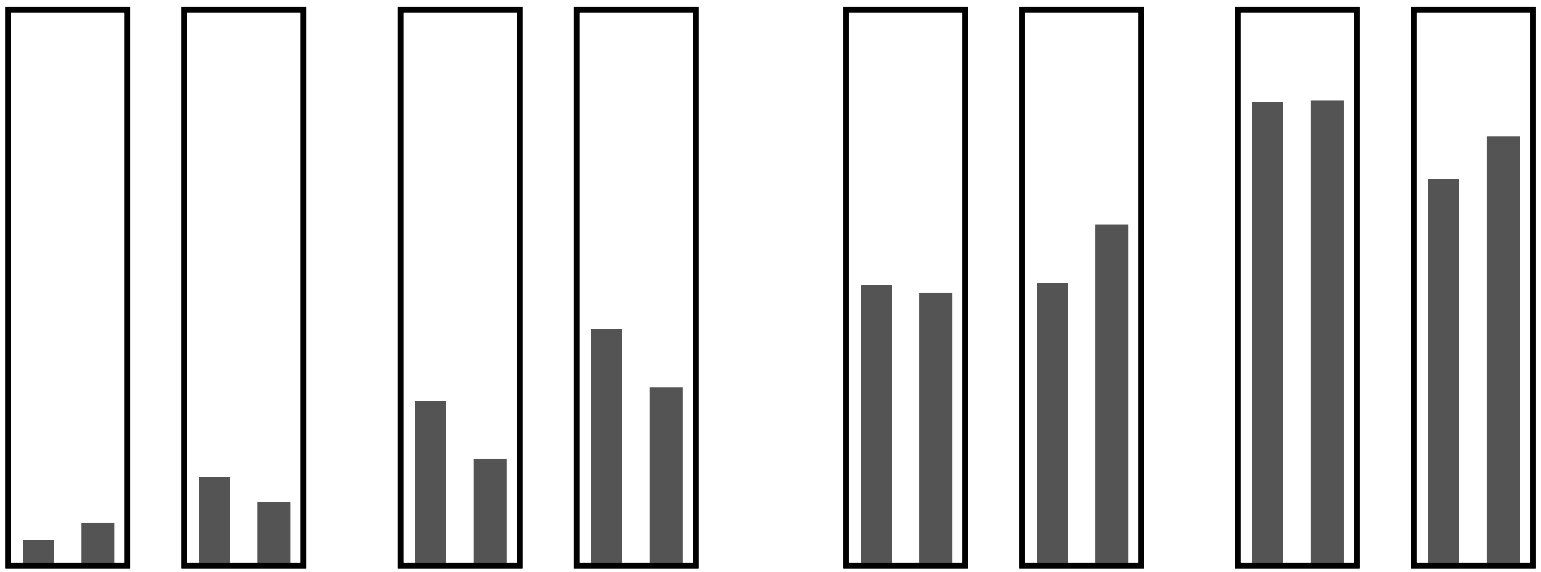

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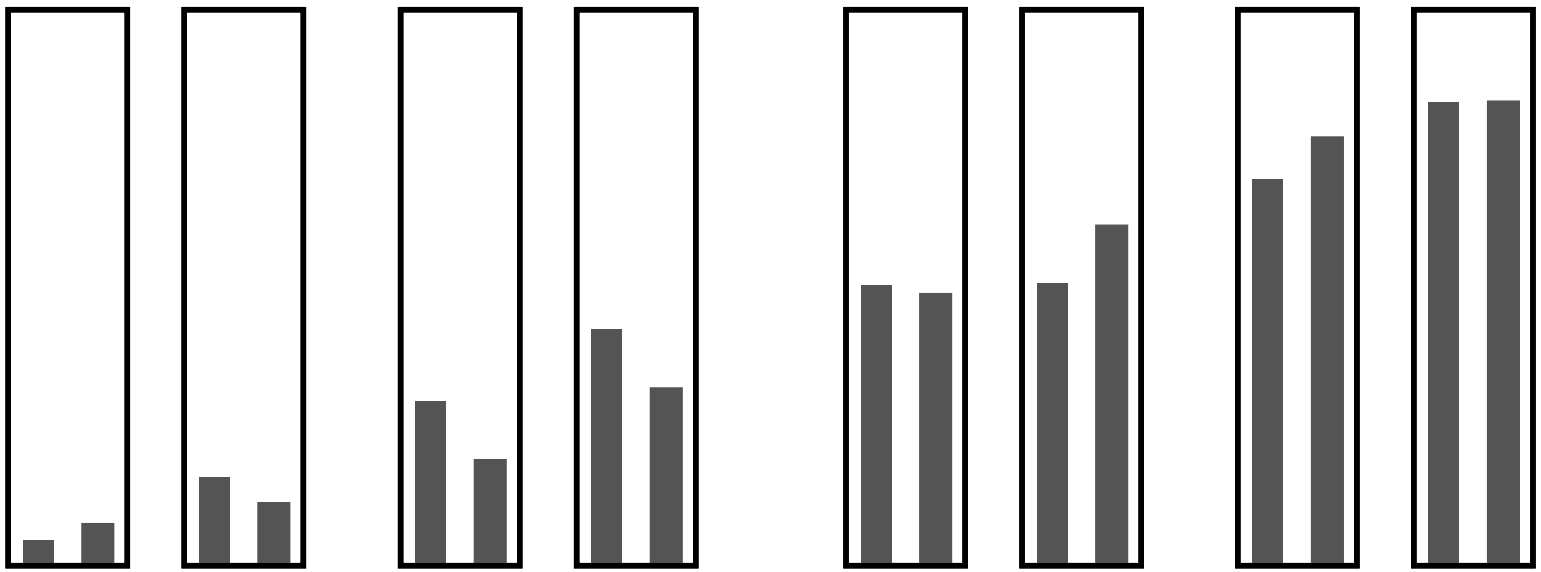

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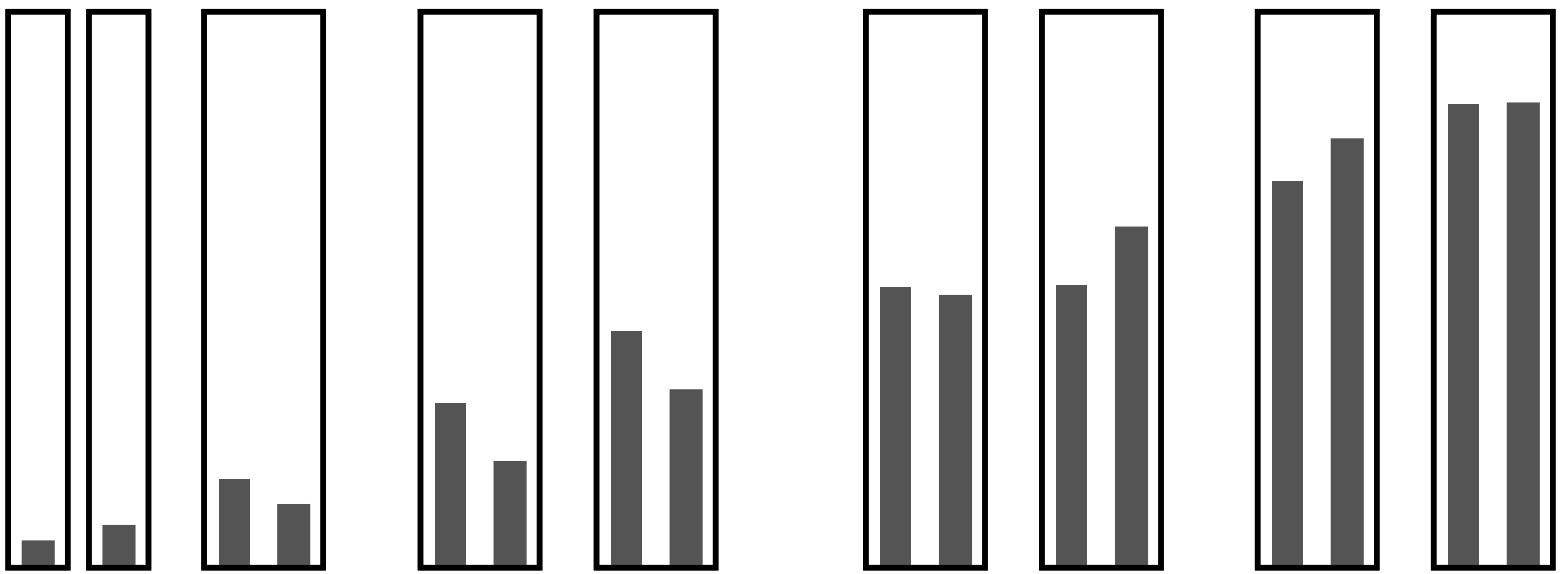

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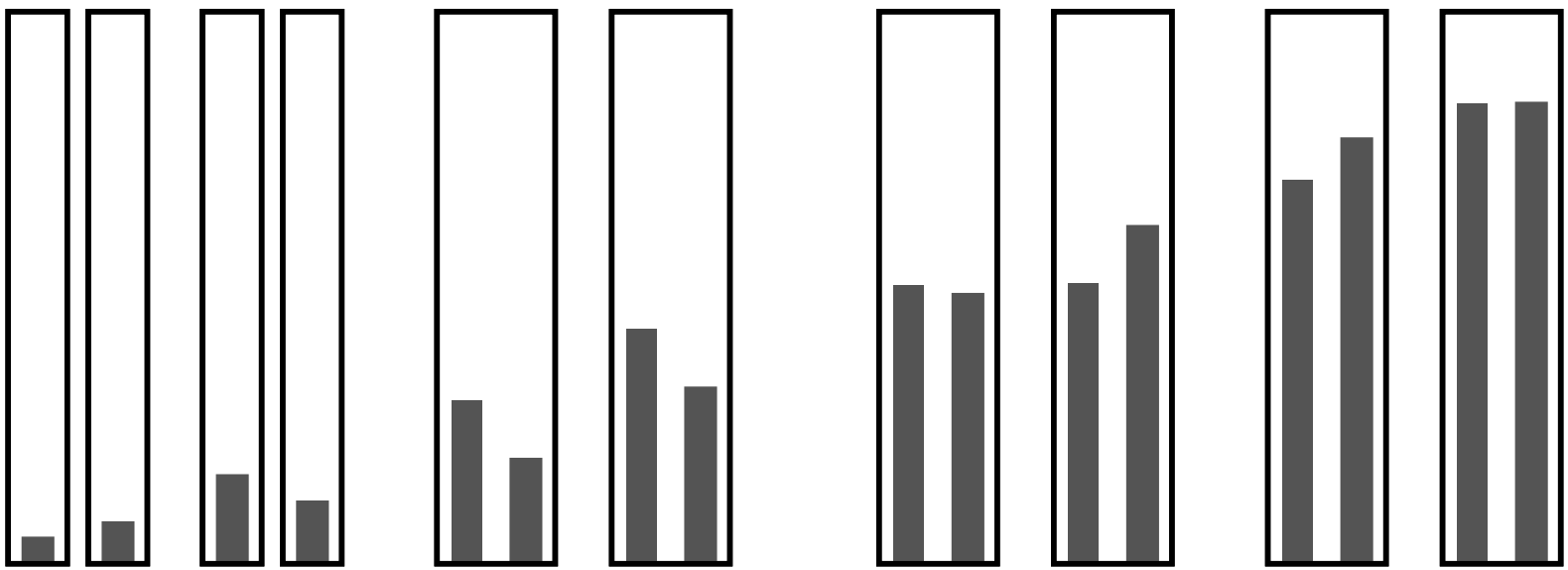

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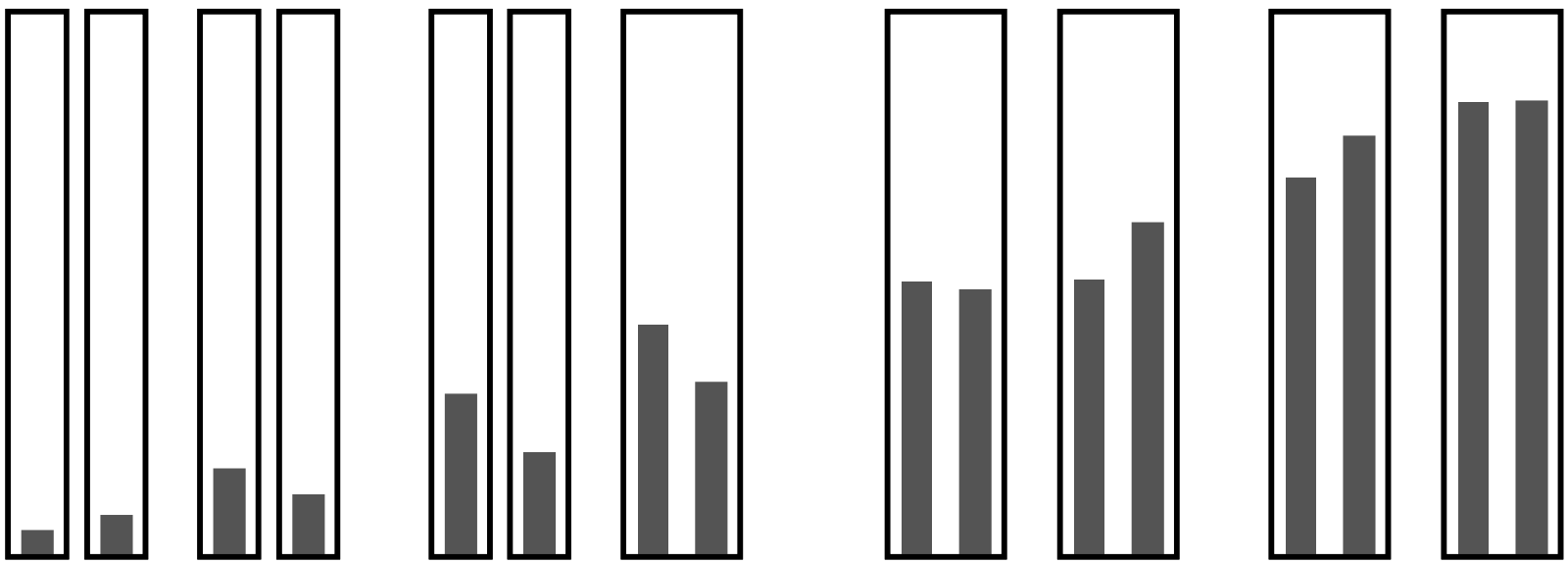

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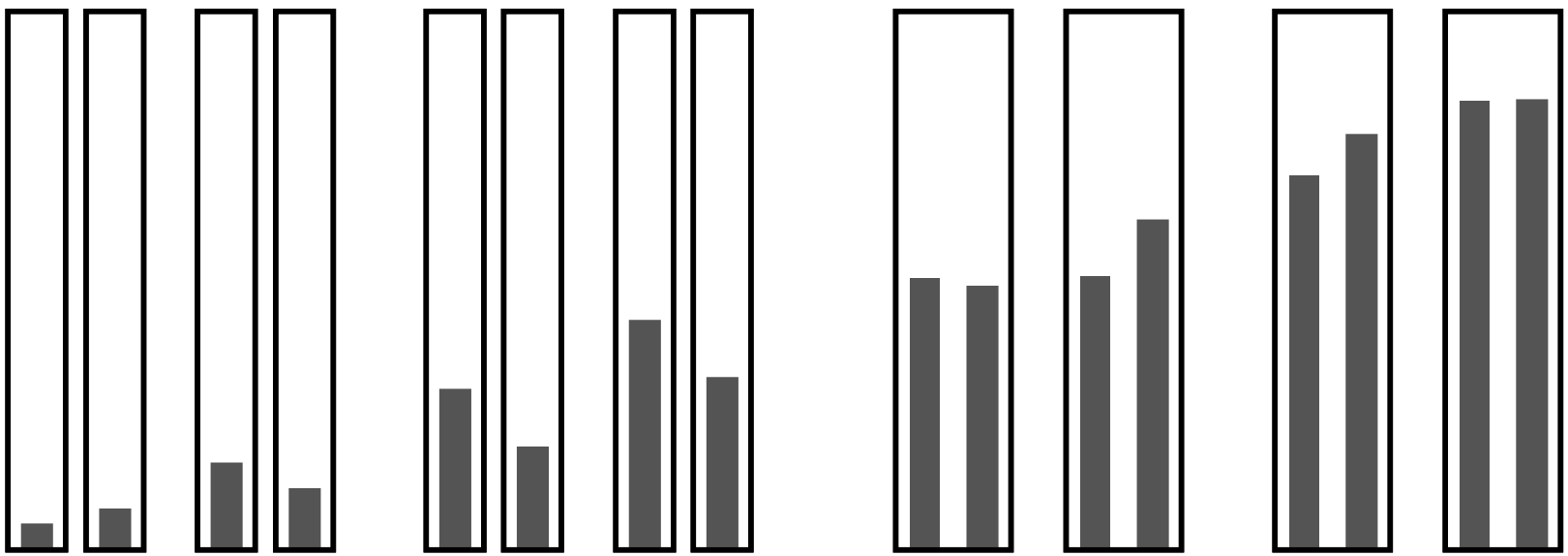

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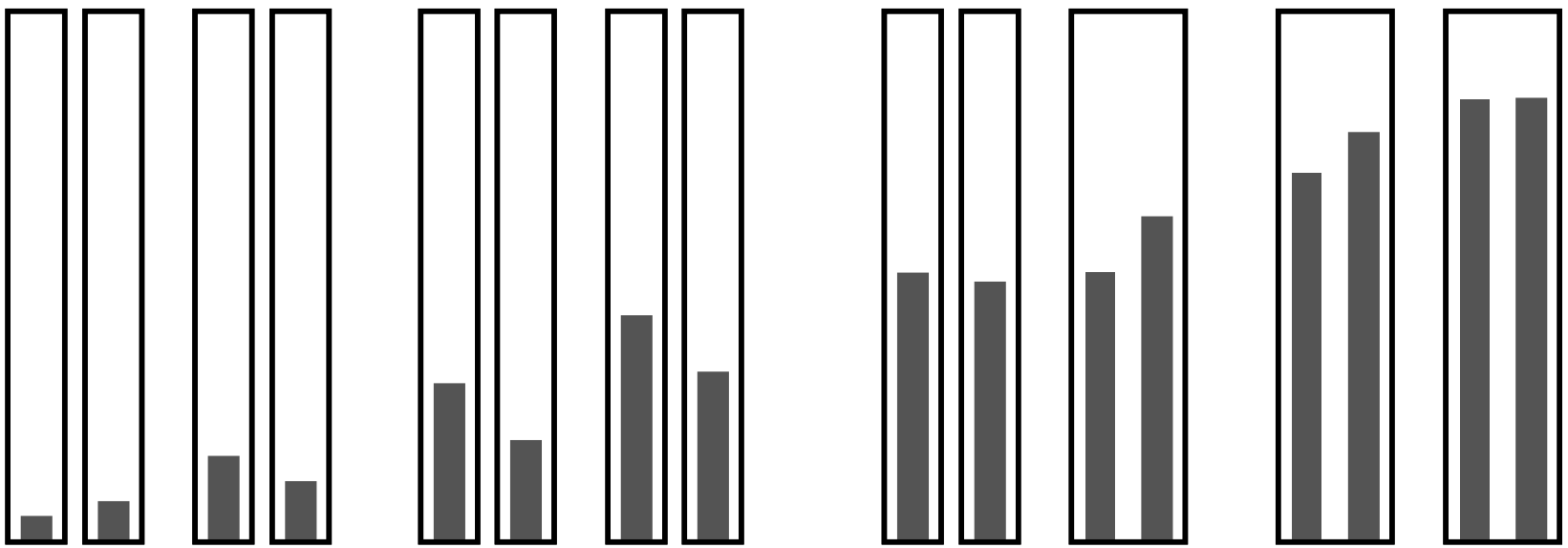

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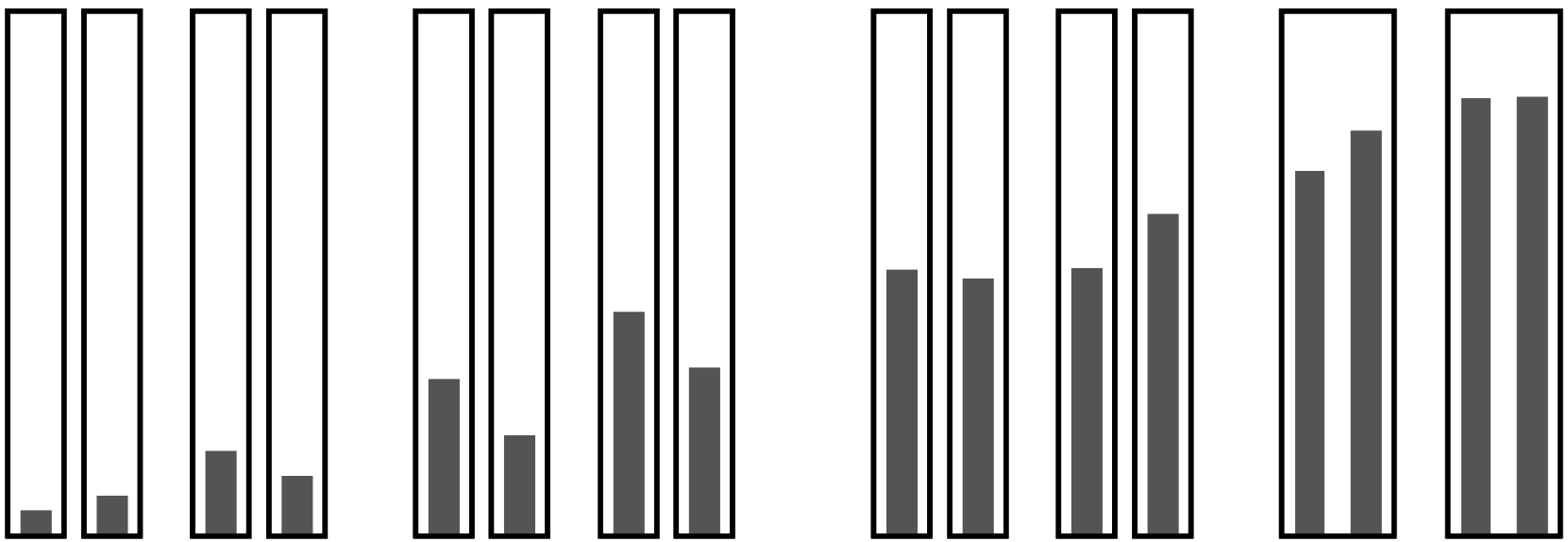

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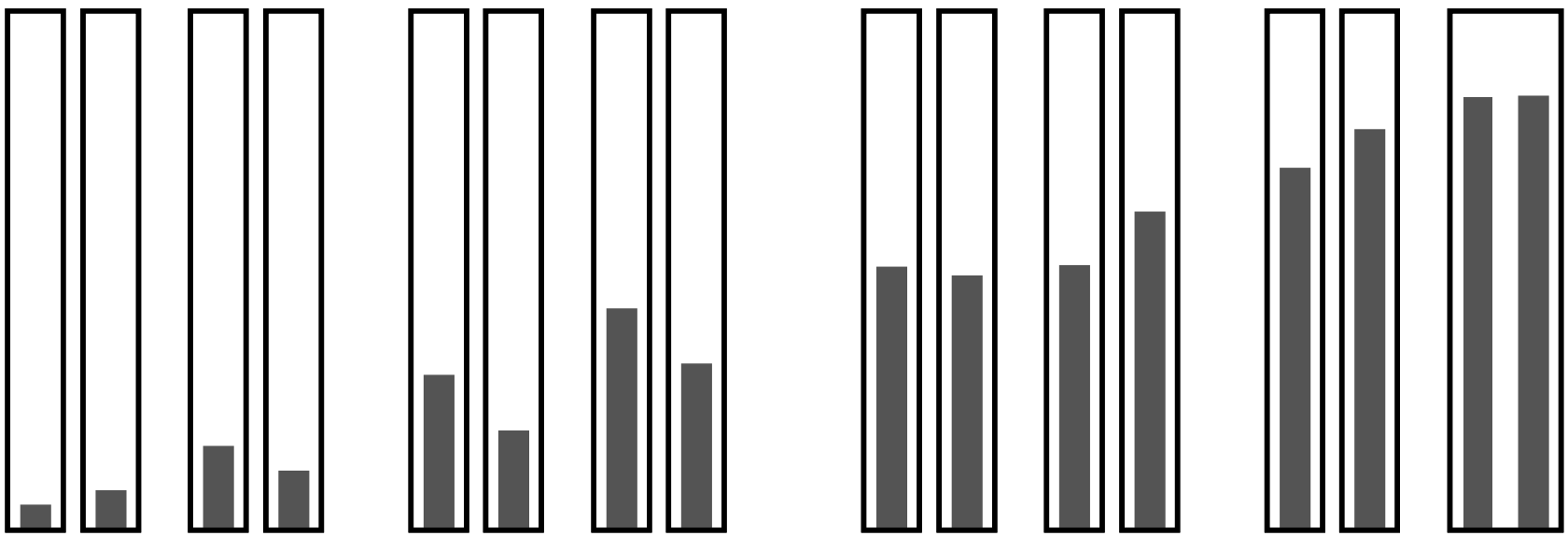

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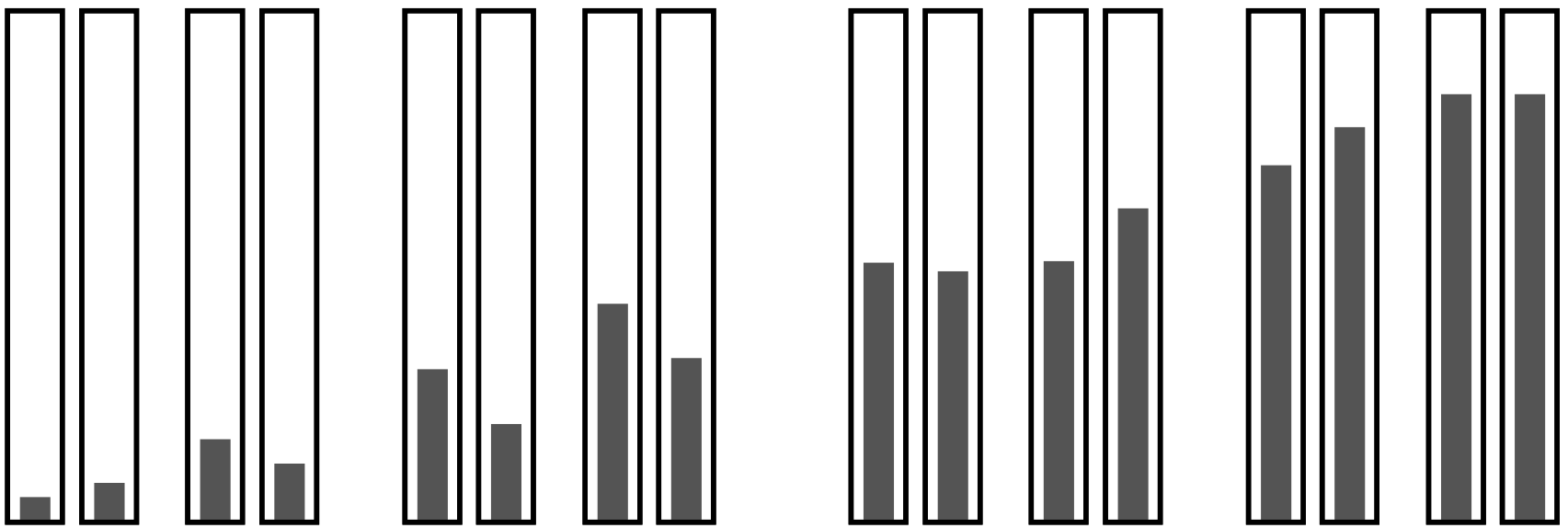

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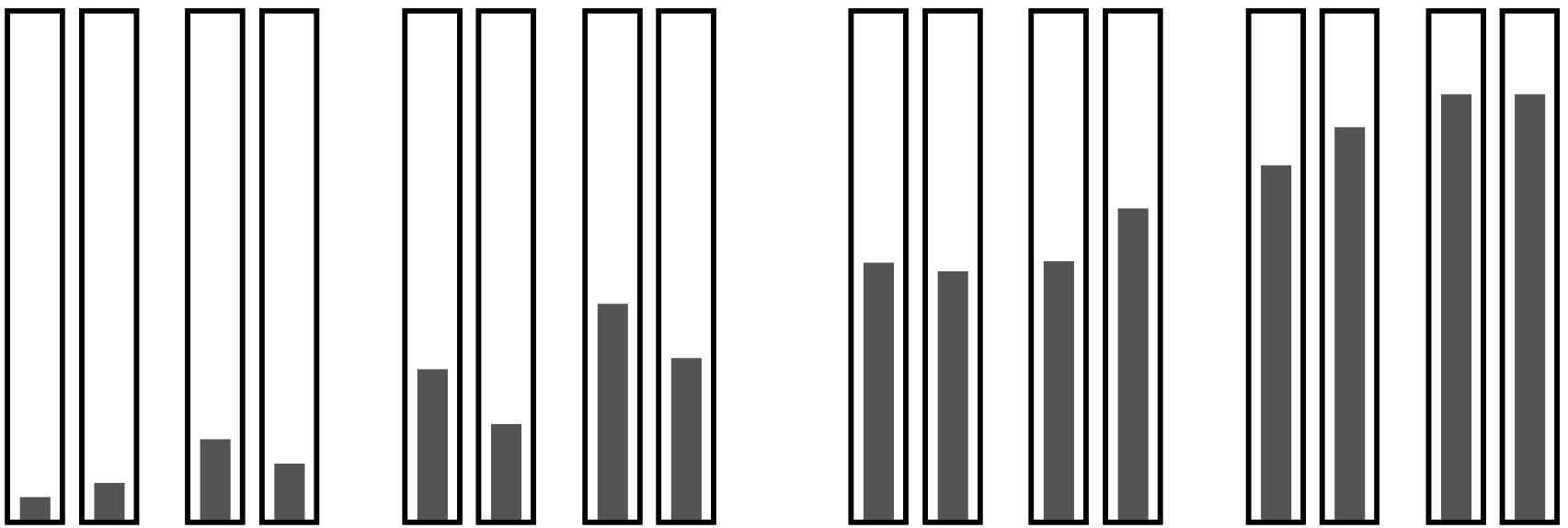

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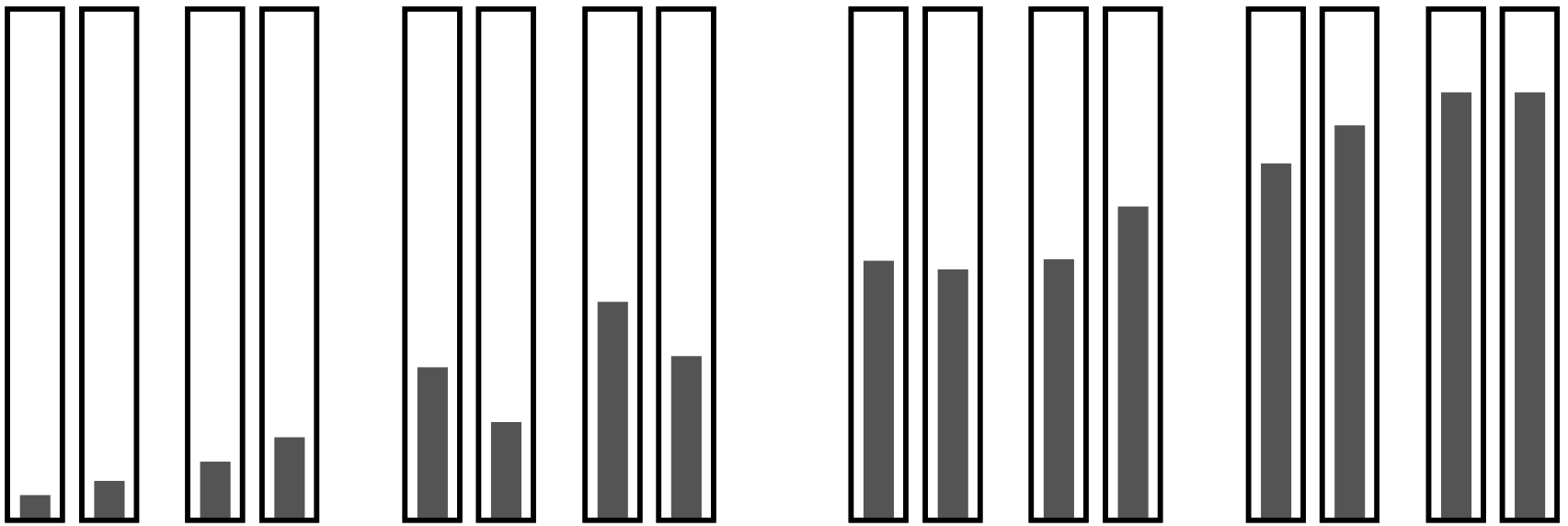

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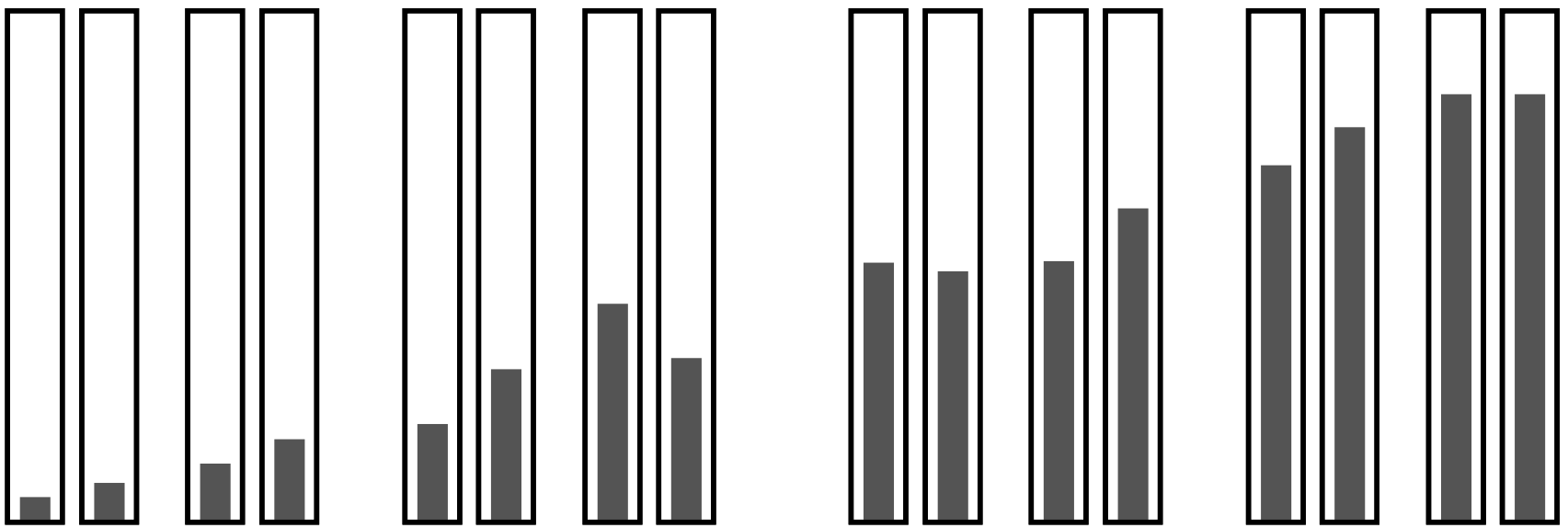

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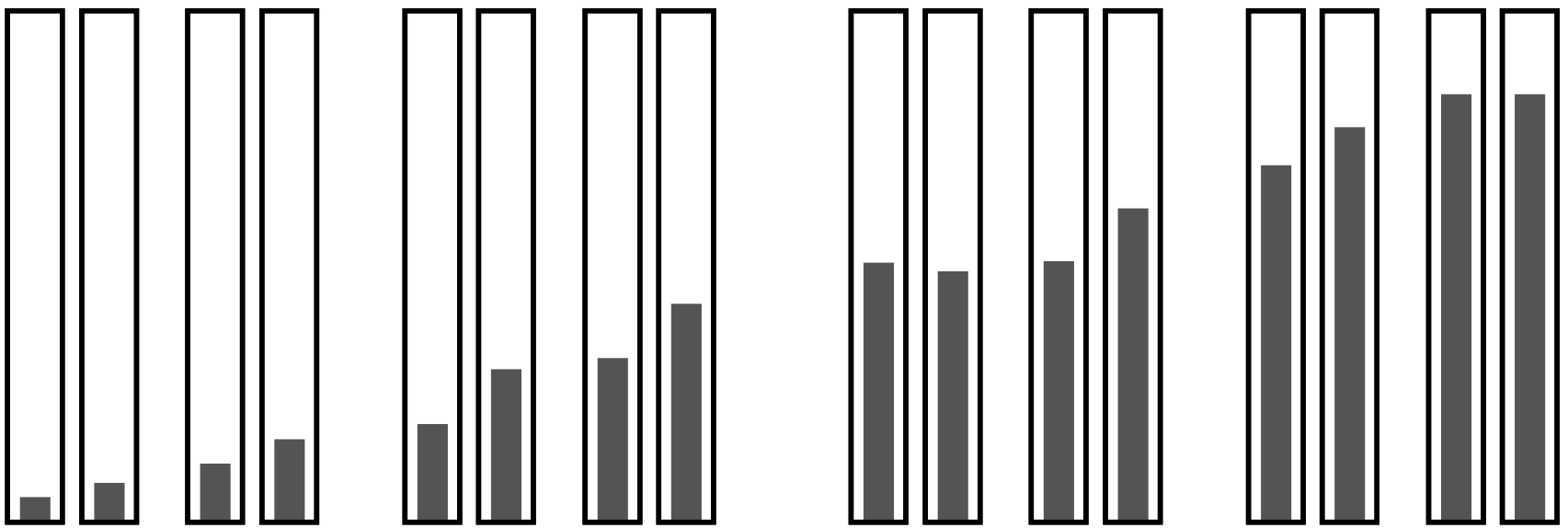

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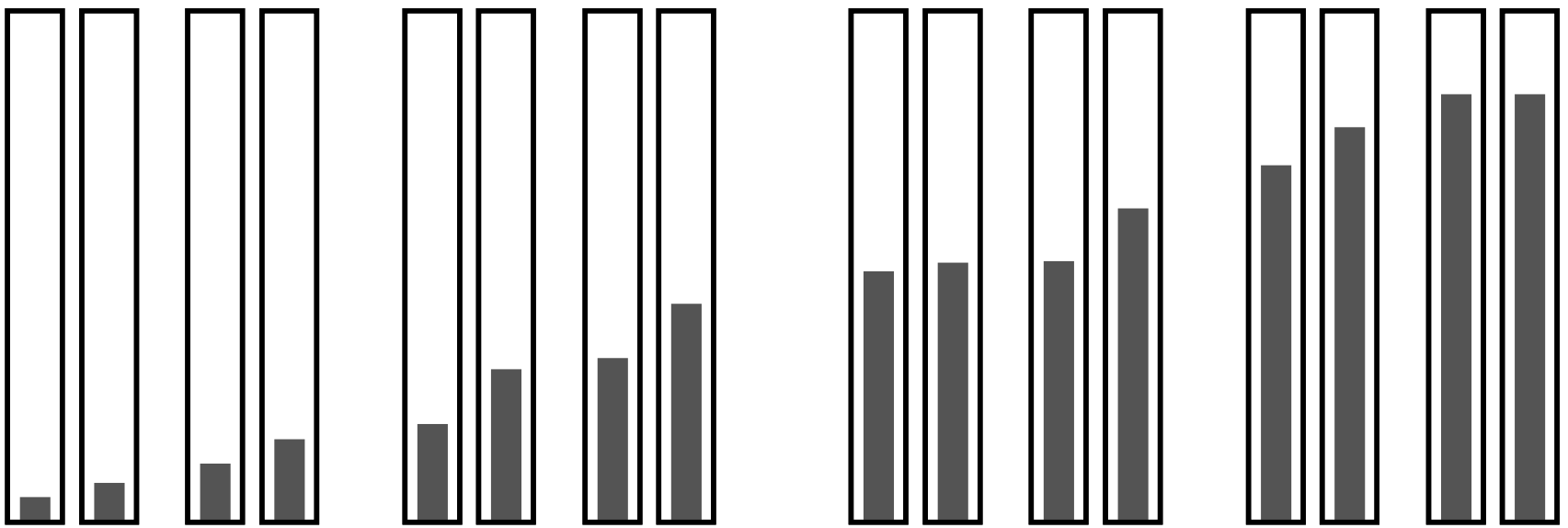

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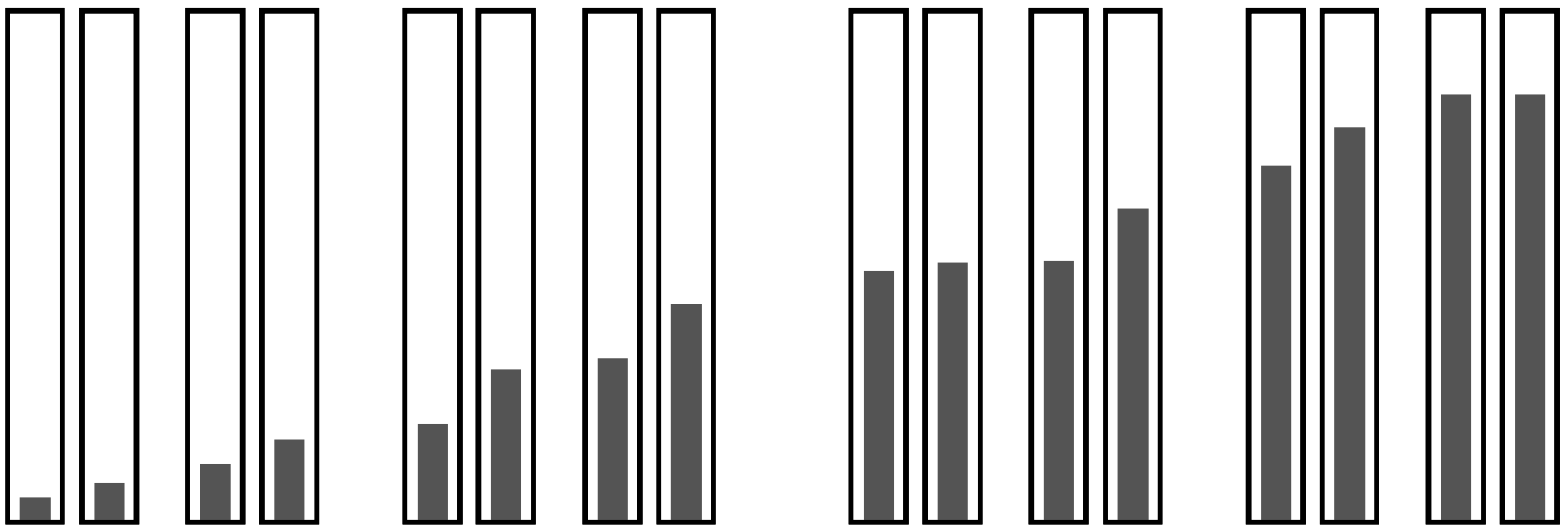

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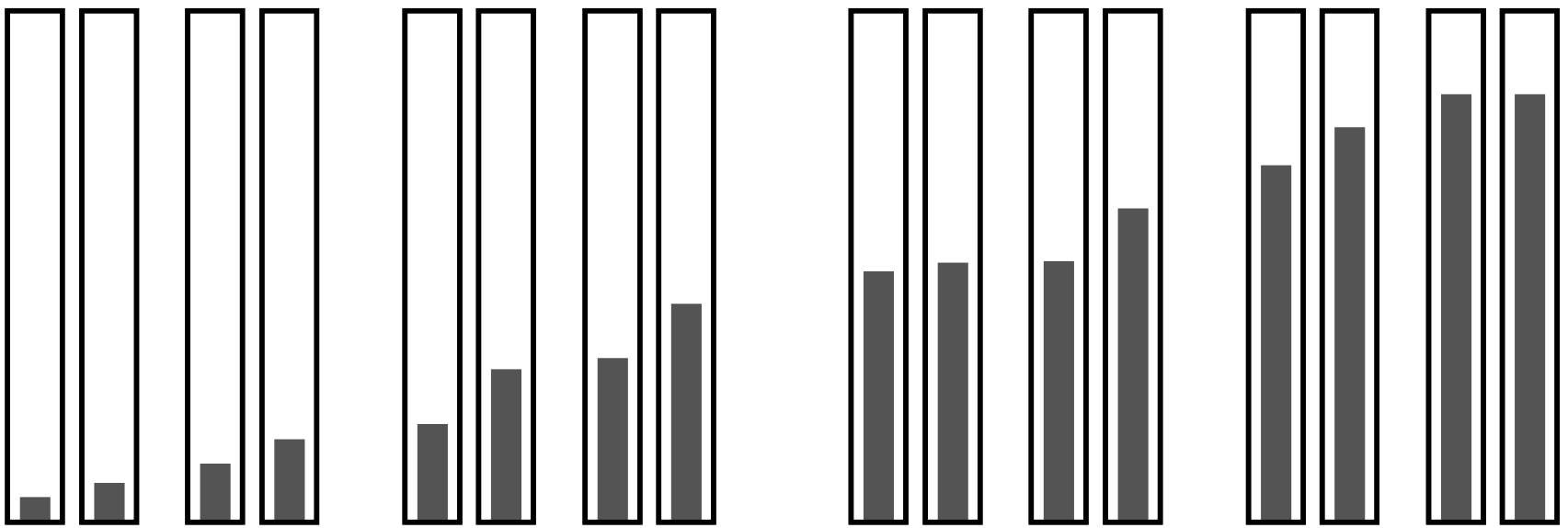

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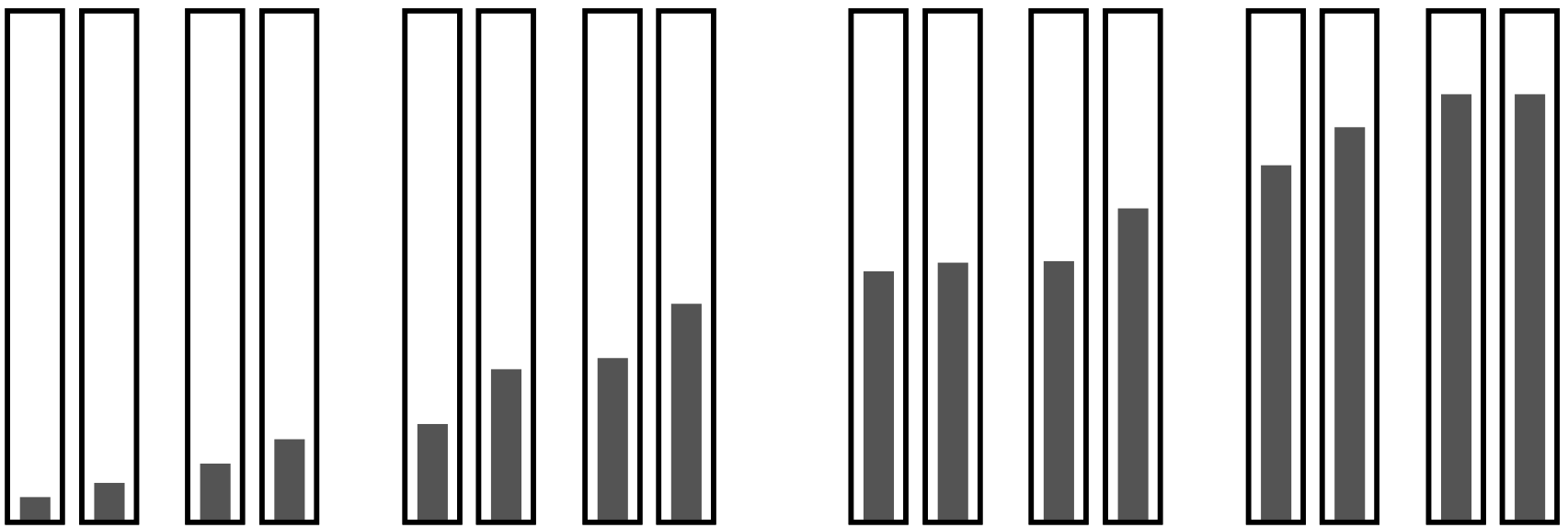

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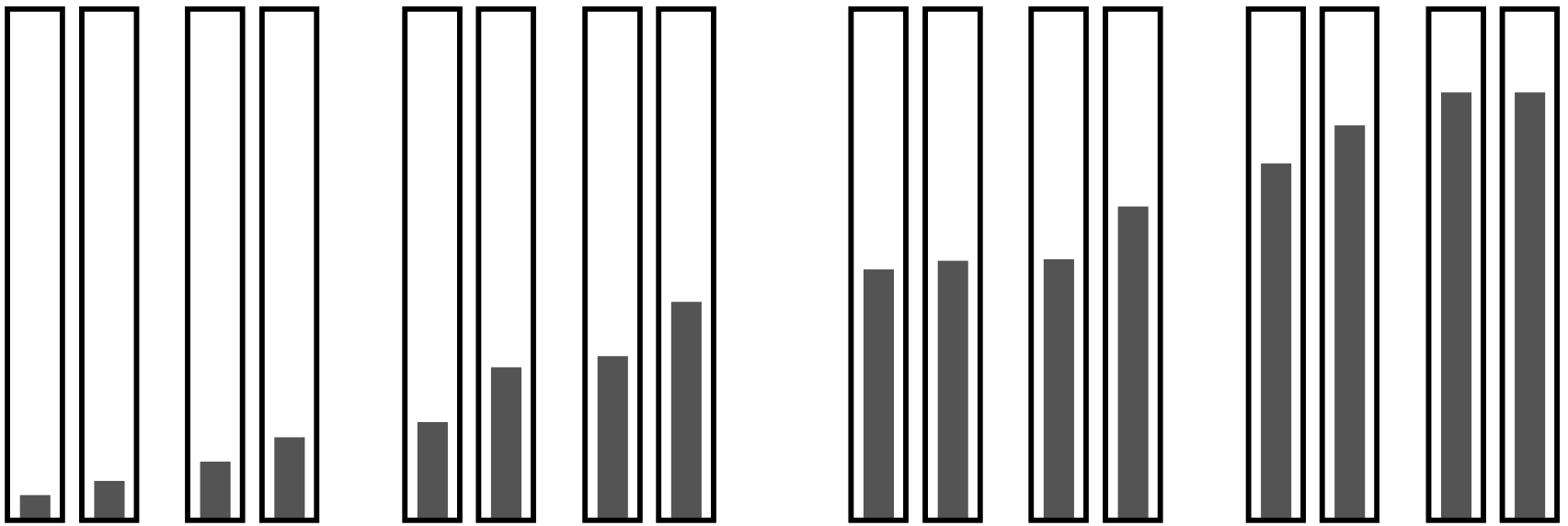

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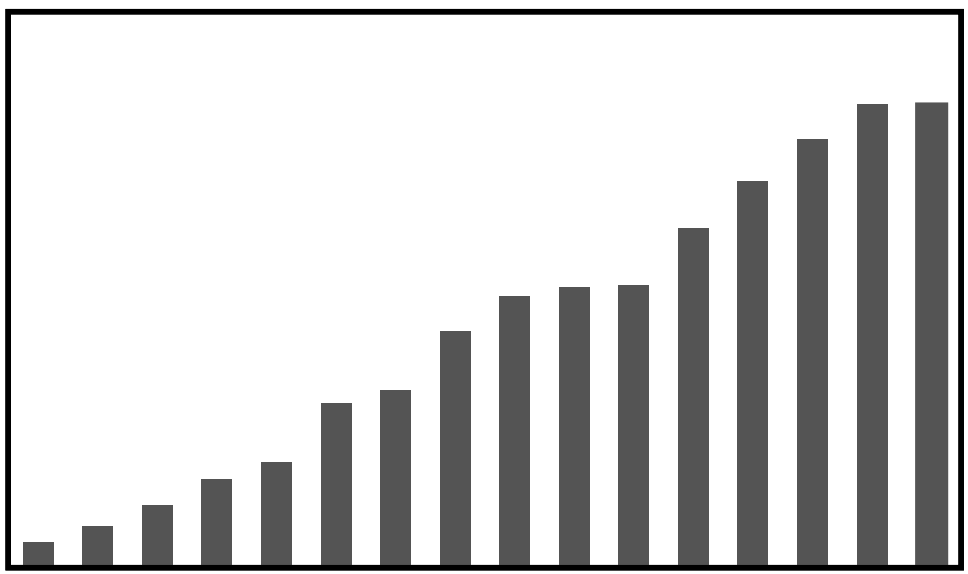

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- complexity is $O(n \cdot \log(n)^2)$
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- correctness is not obvious

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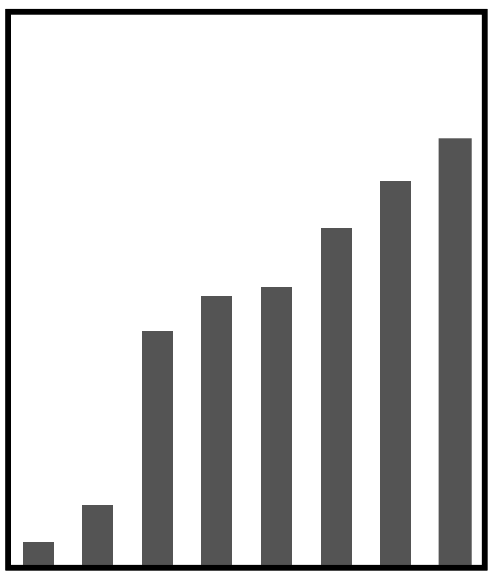

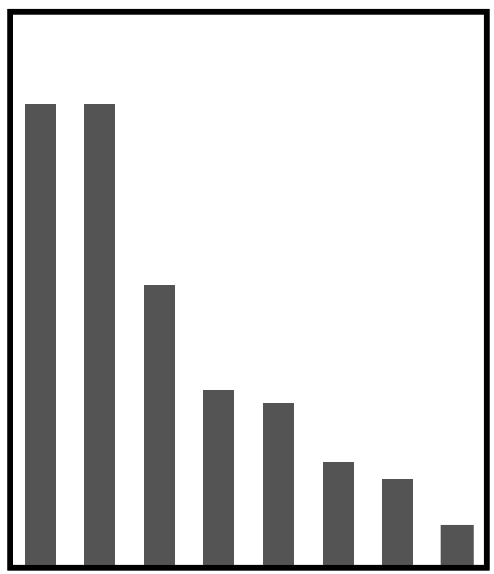

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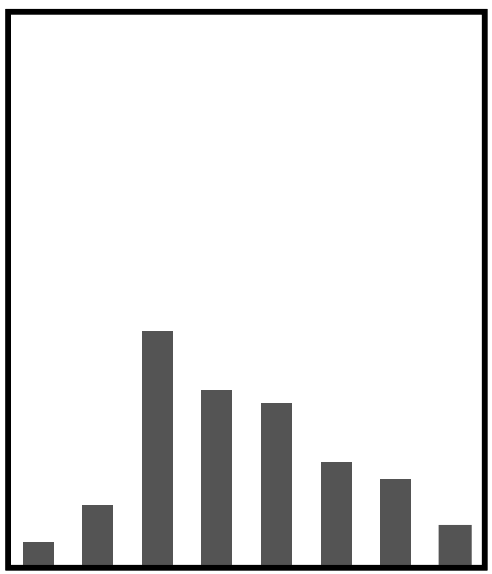

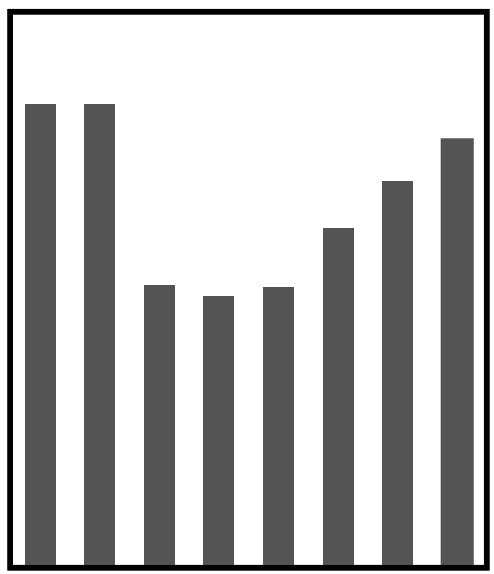

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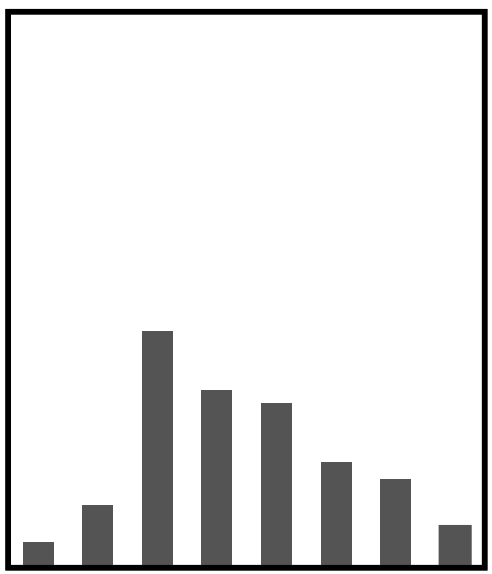

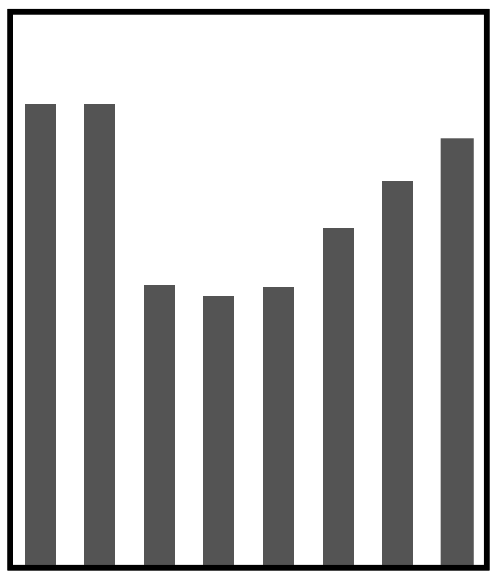

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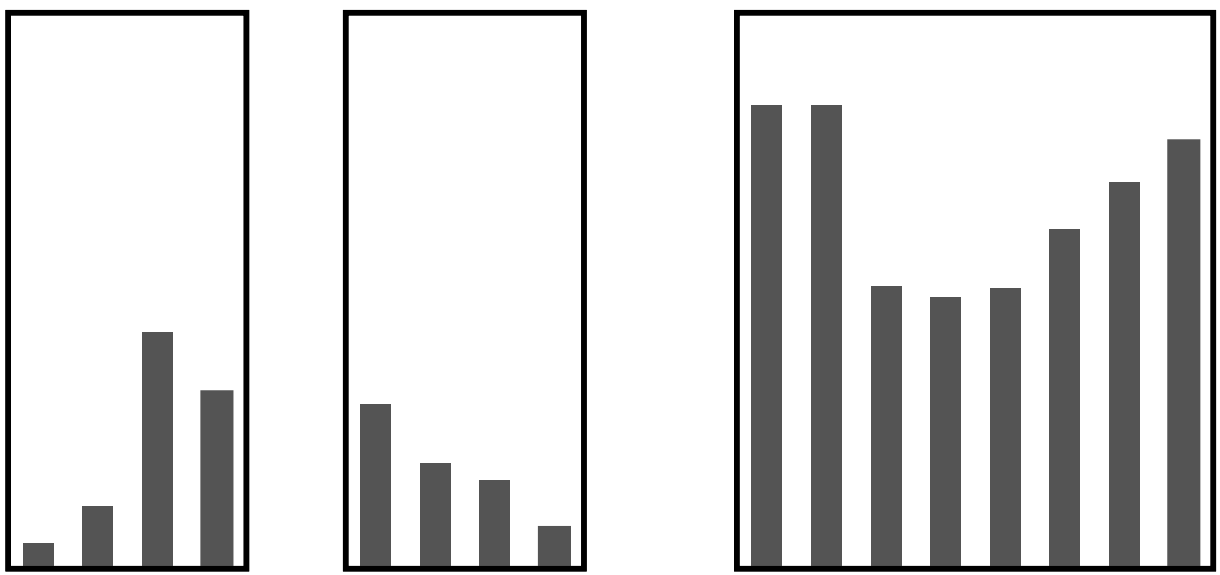

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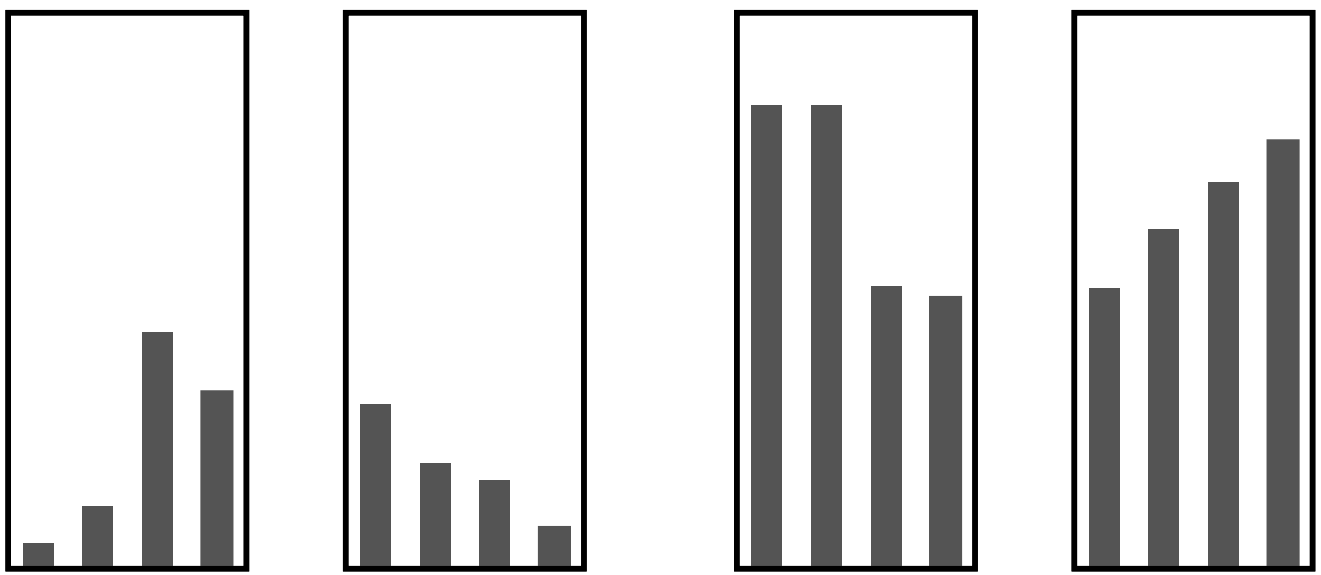

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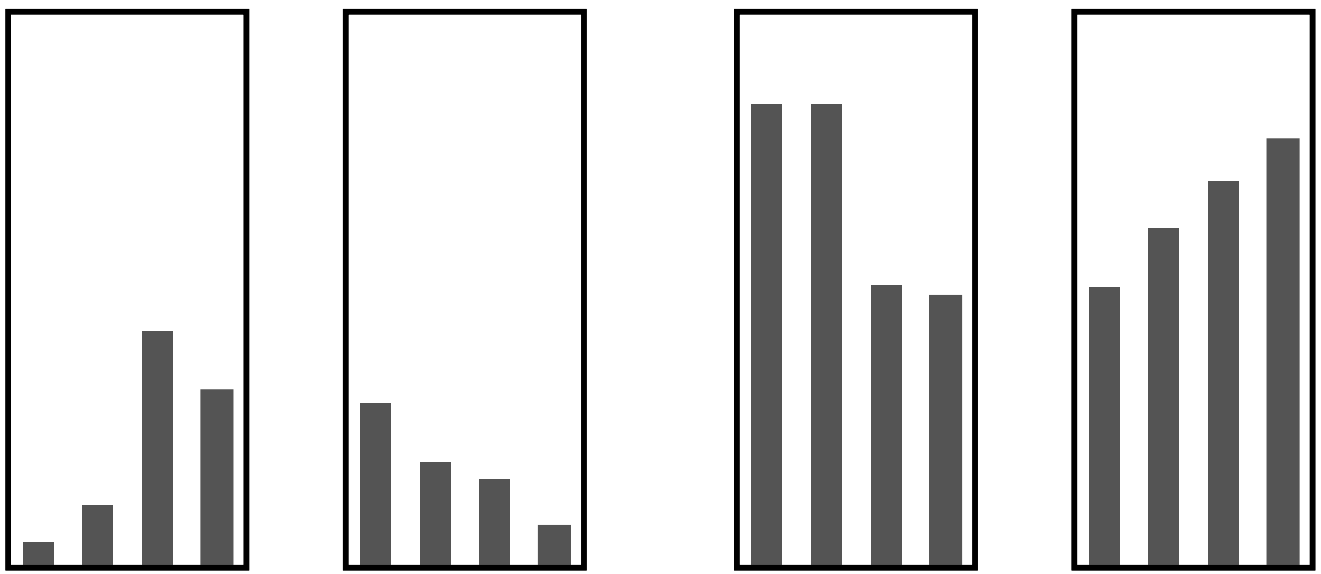

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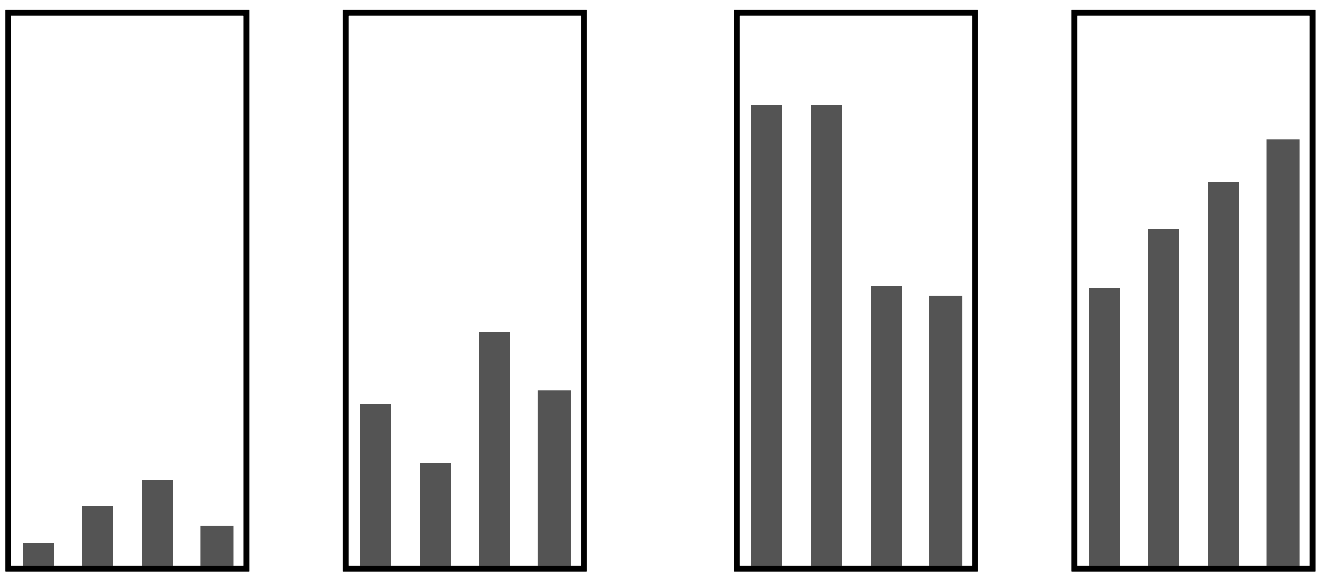

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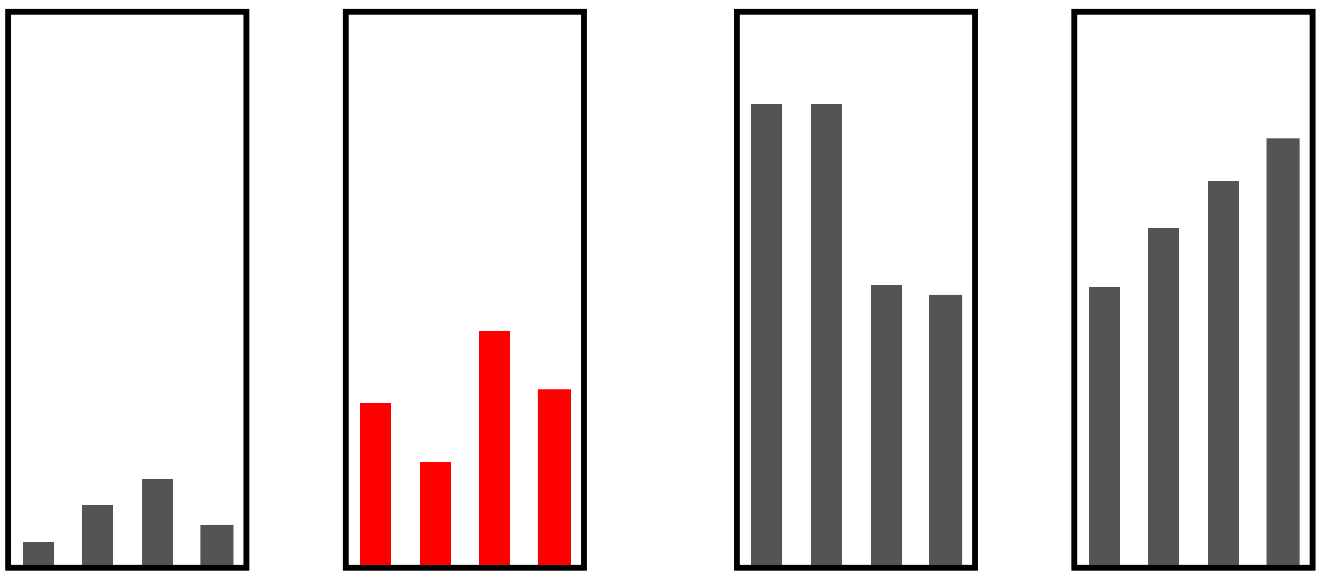

Knuth's 0-1-Principle [Knuth 1973]

Informally: If a comparison-swap algorithm sorts Booleans correctly, it sorts integers correctly as well.

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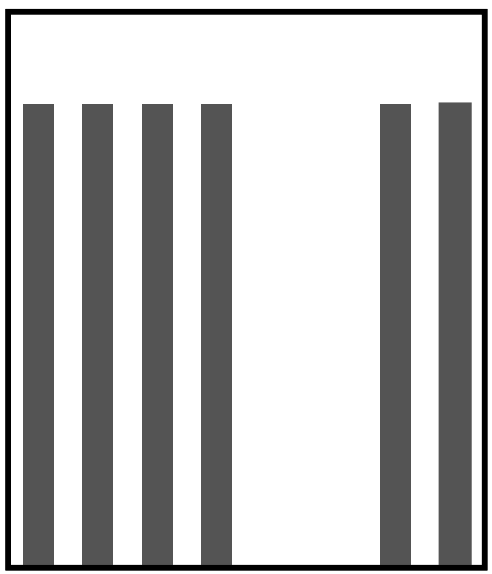

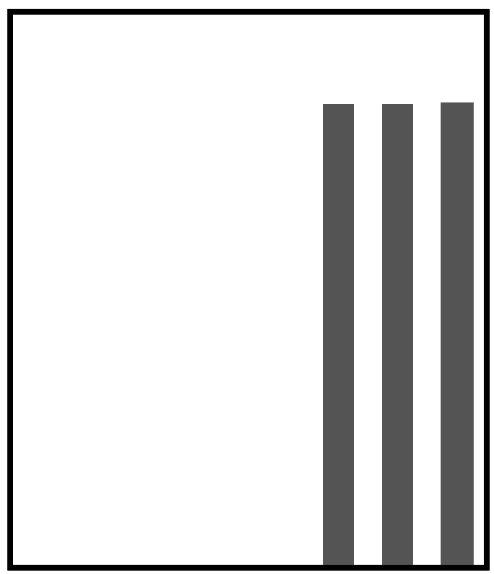

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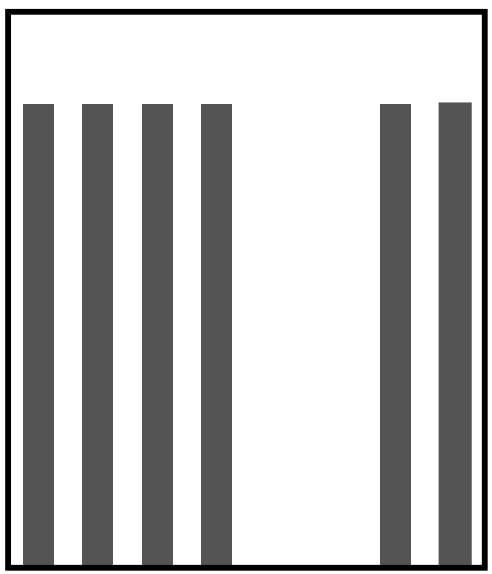

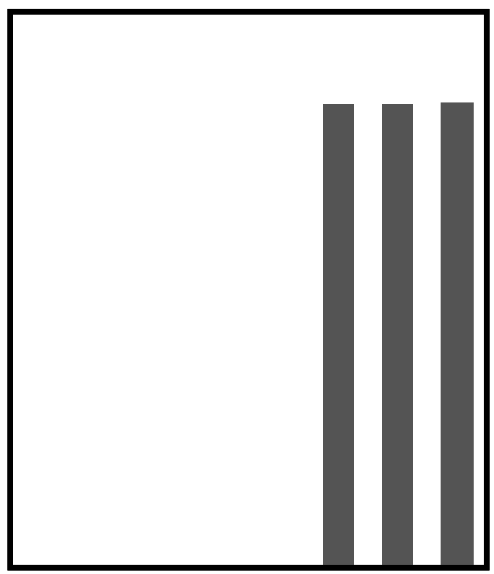

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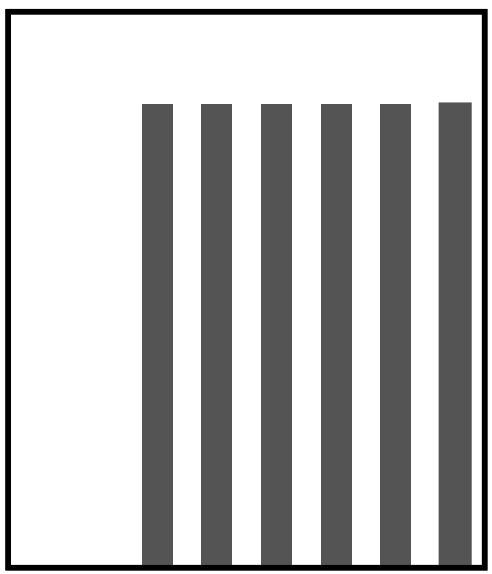

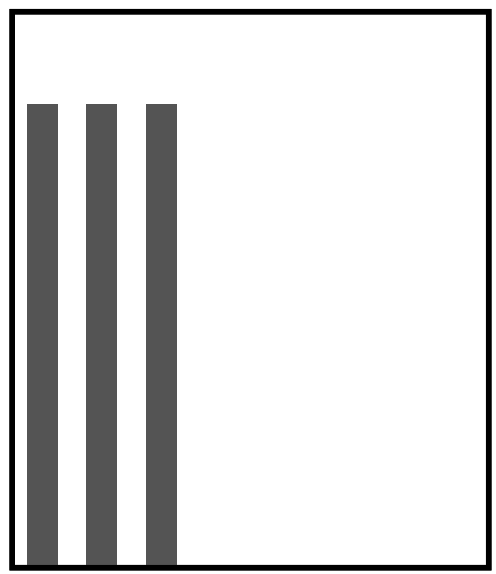

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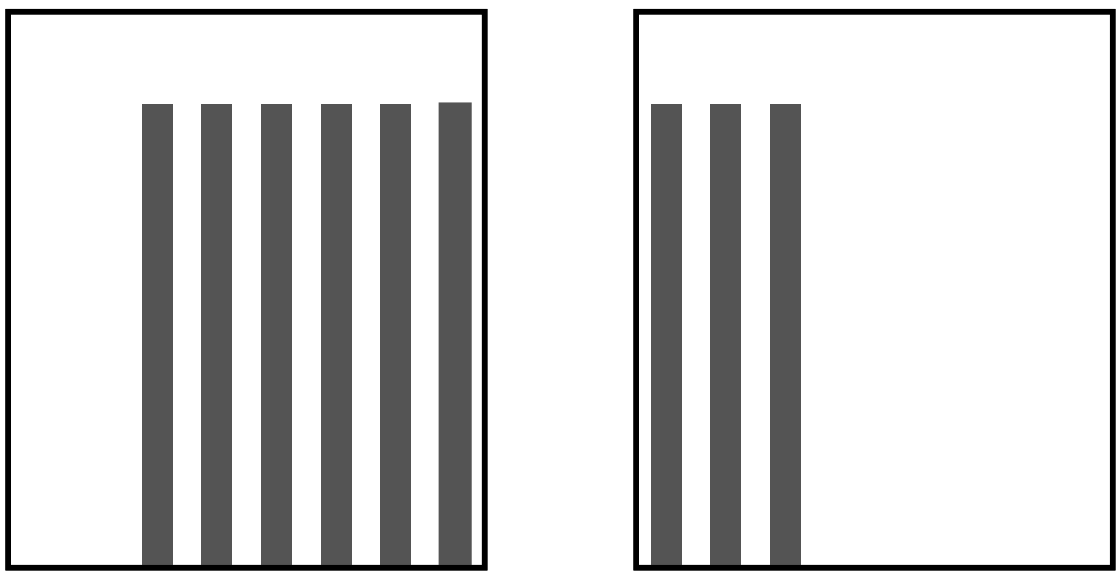

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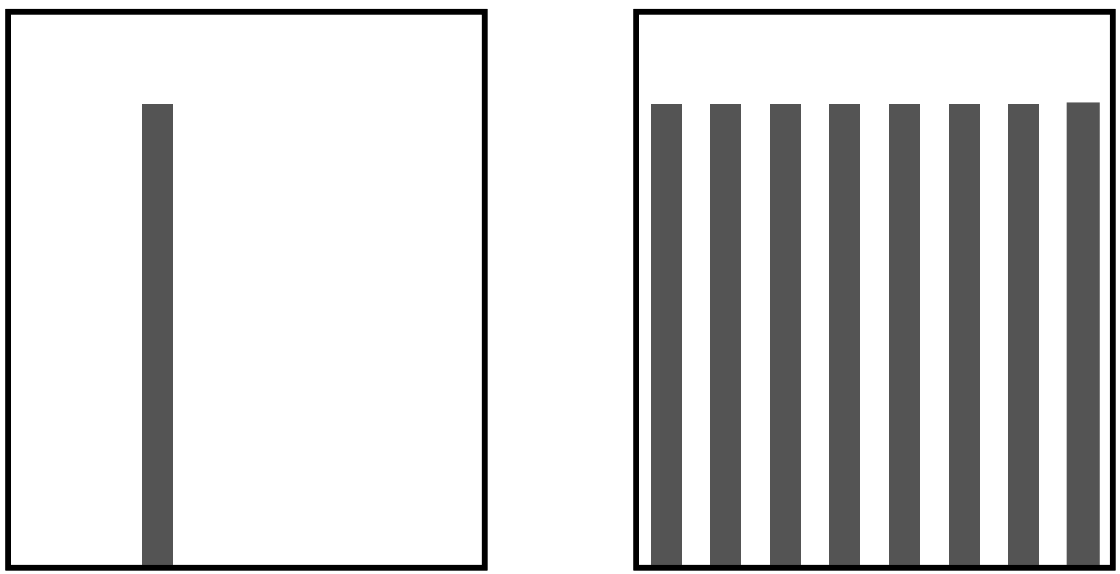

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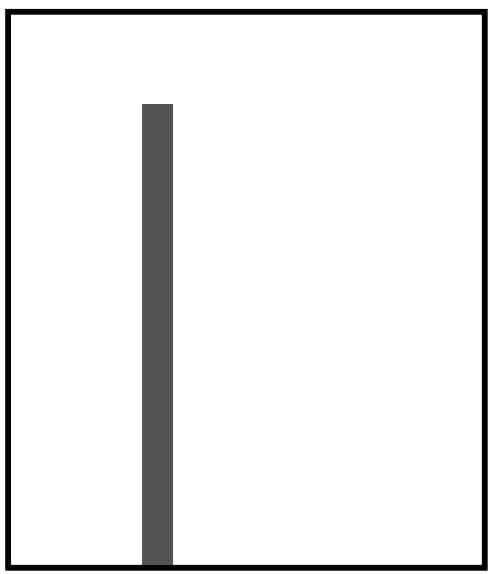

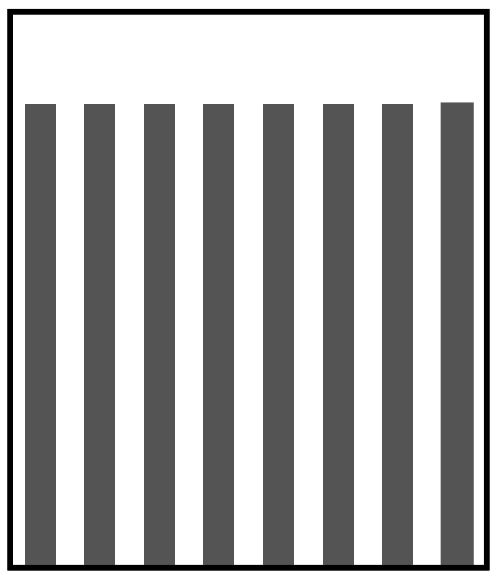

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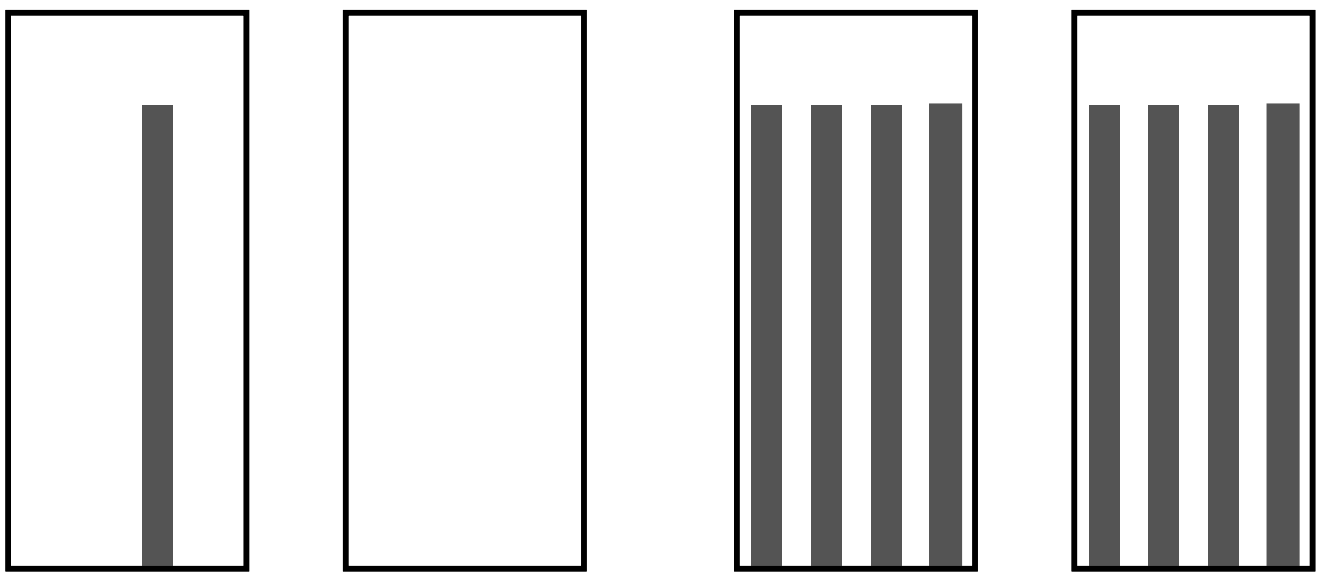

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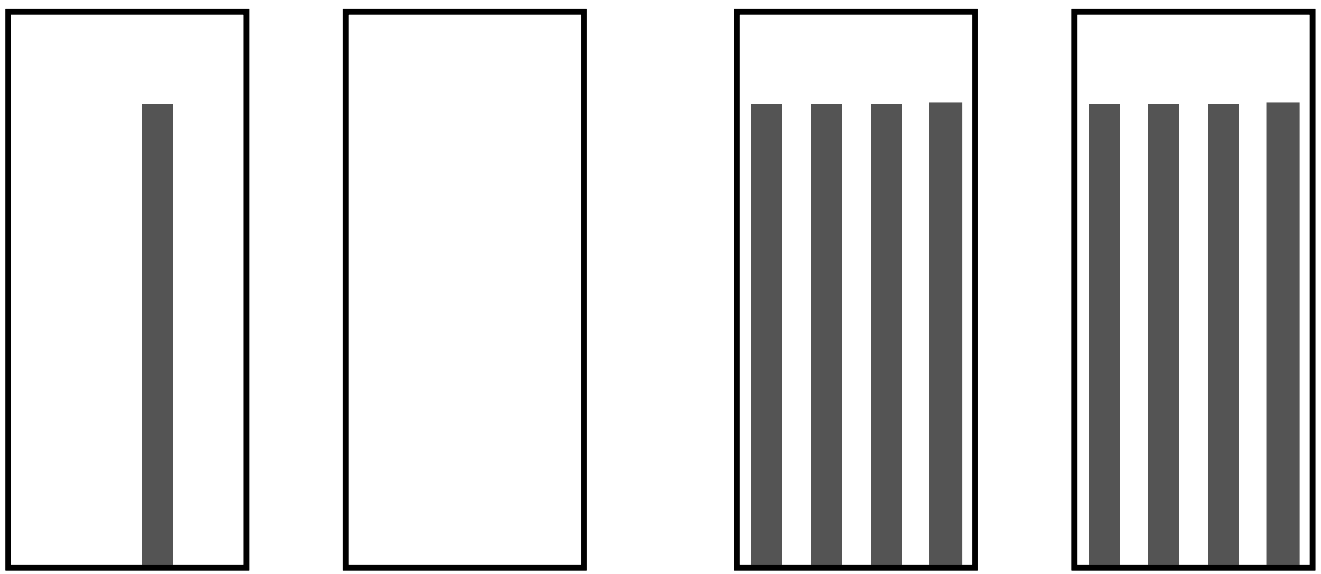

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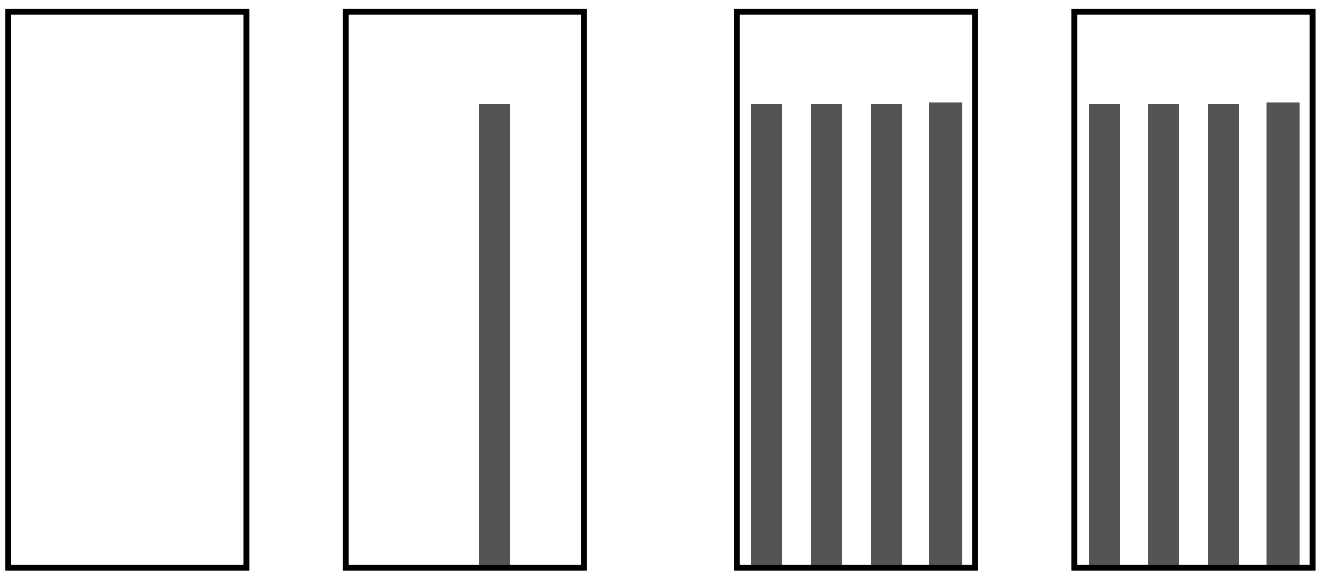

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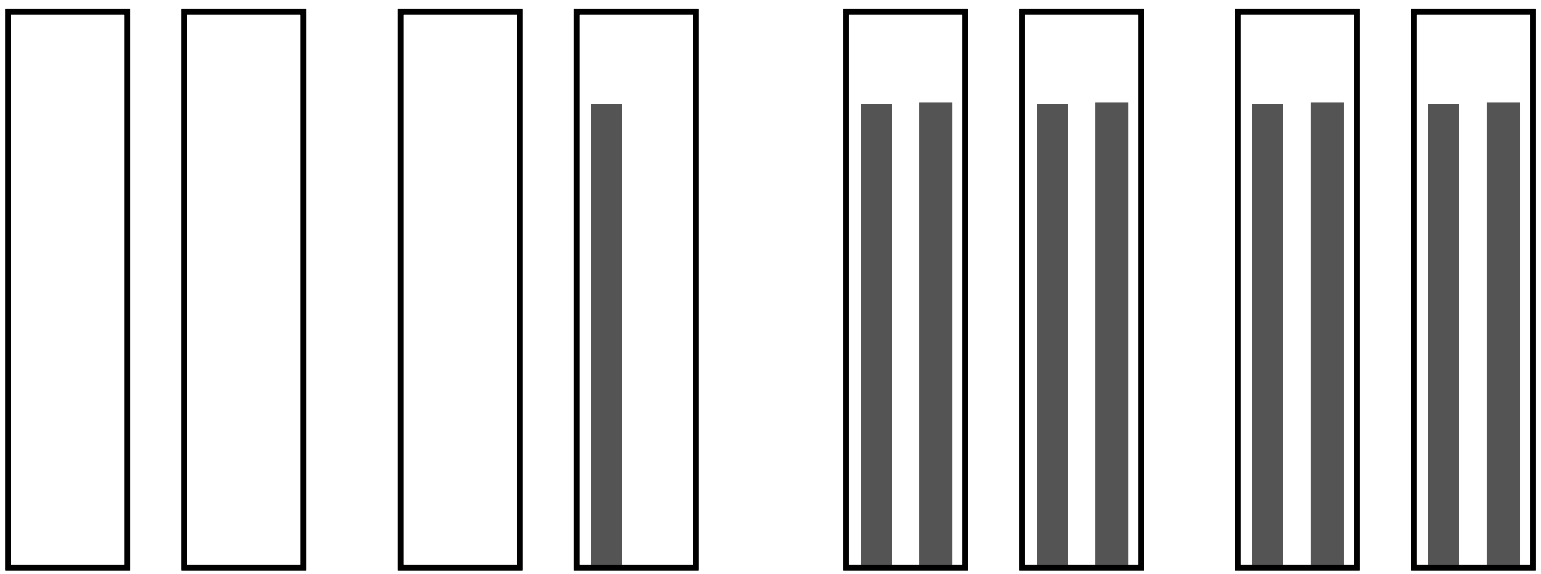

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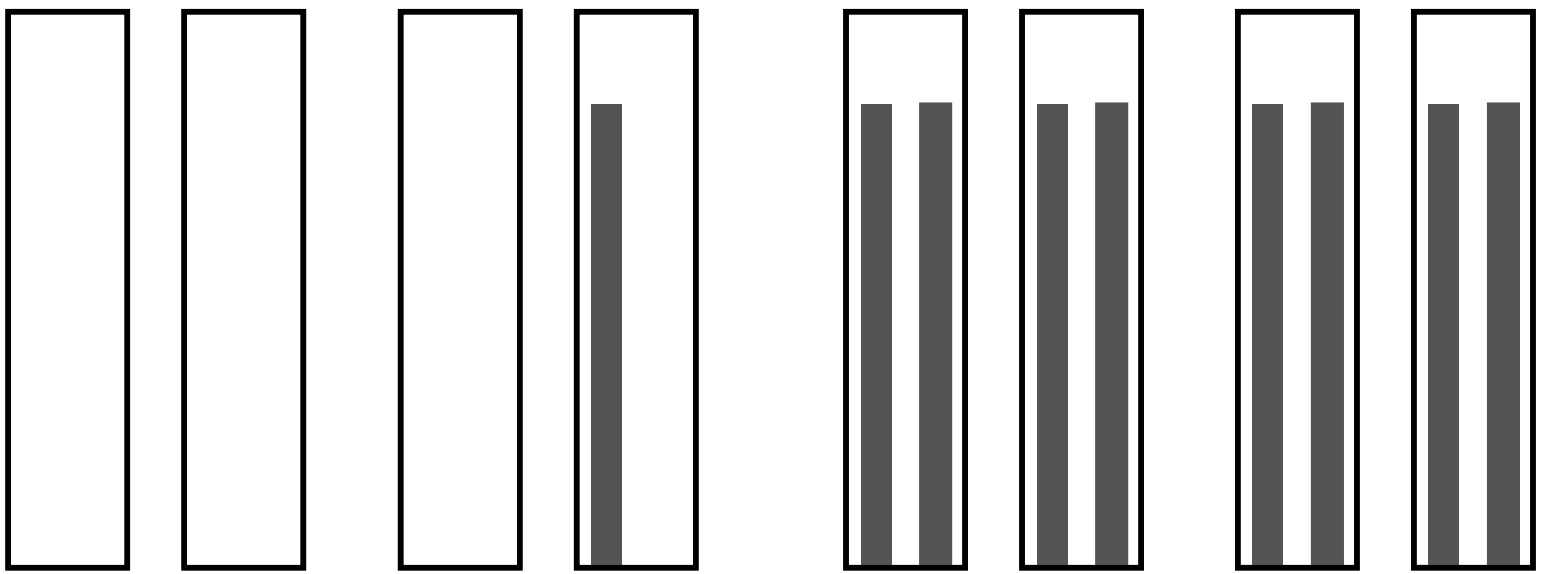

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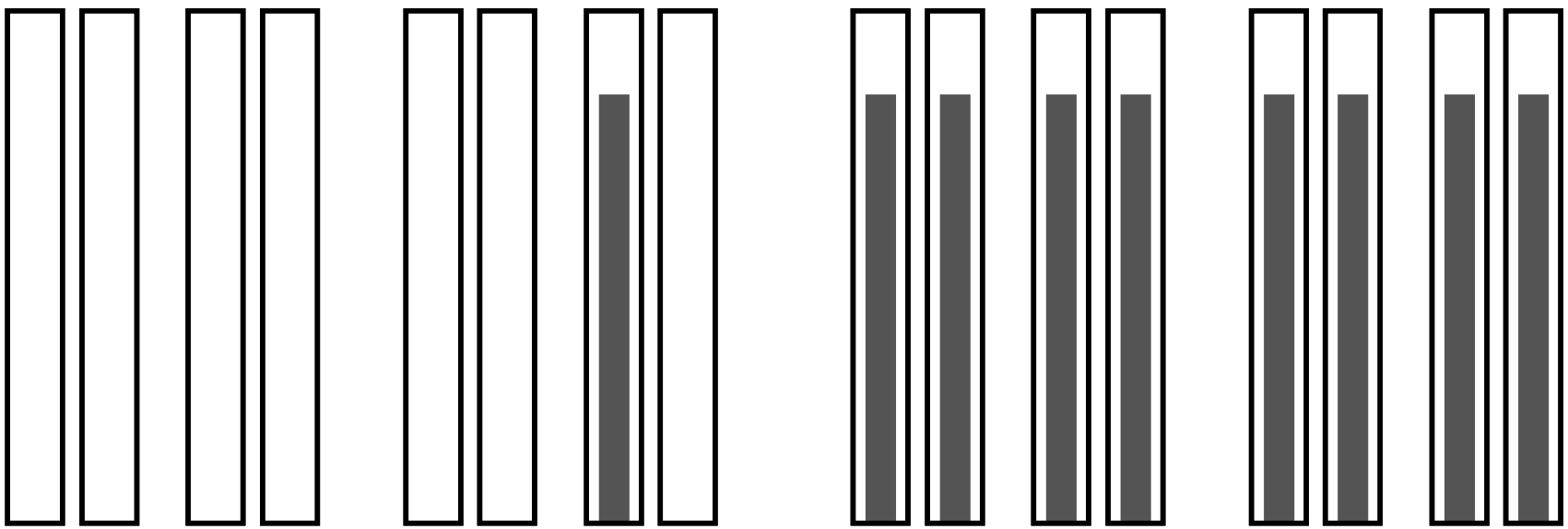

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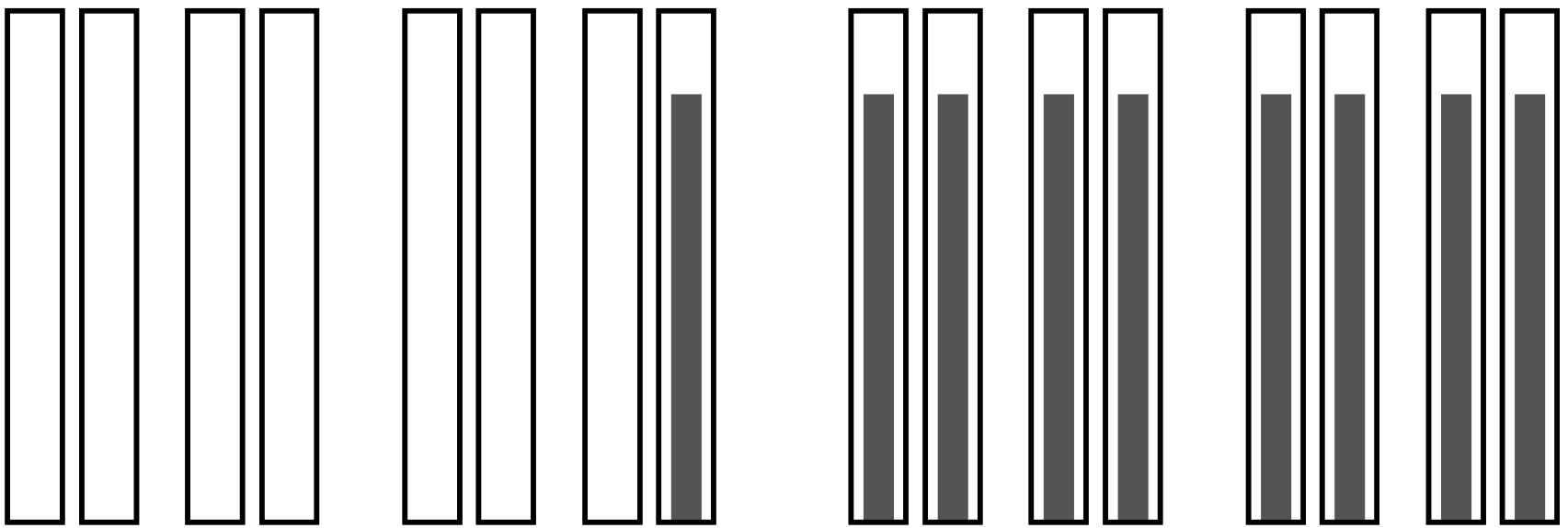

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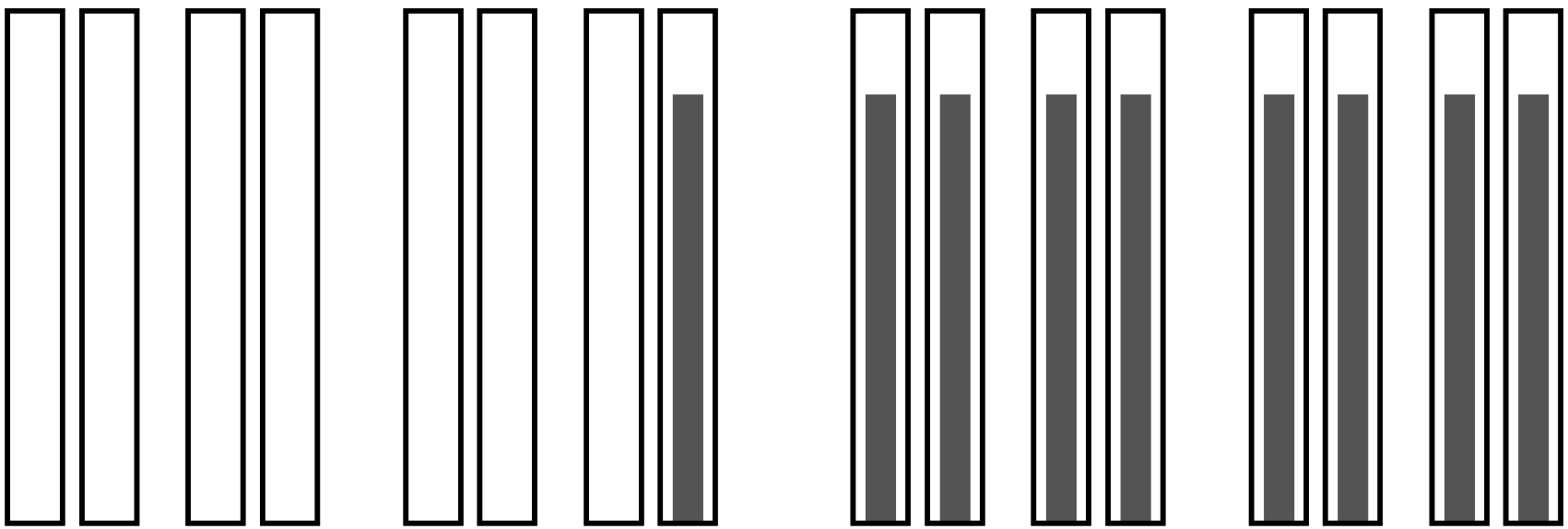

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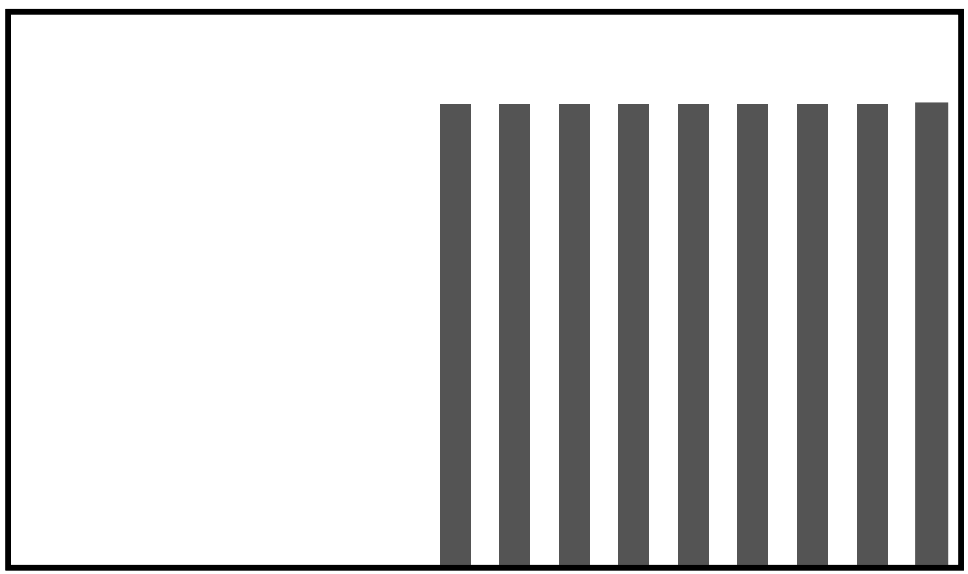

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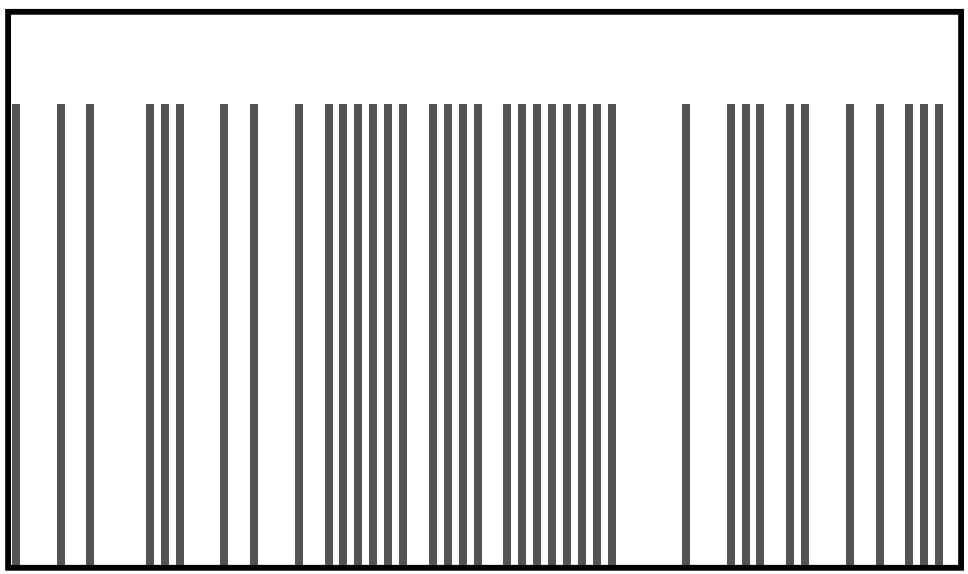

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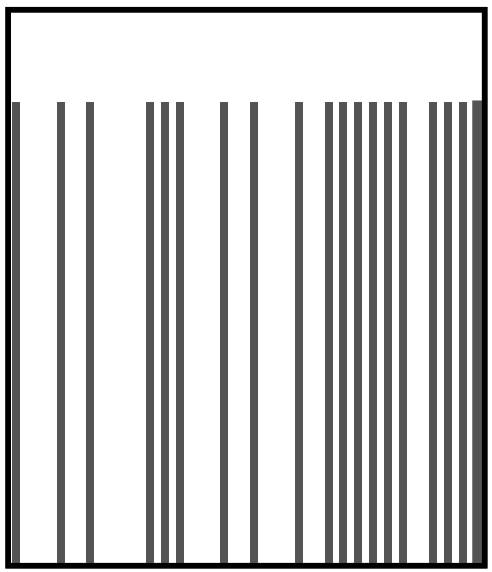

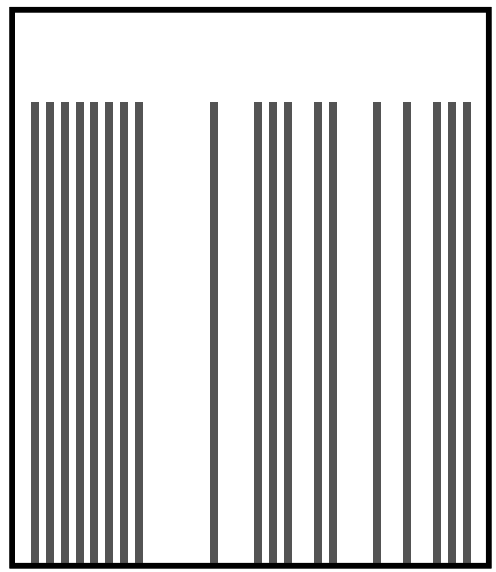

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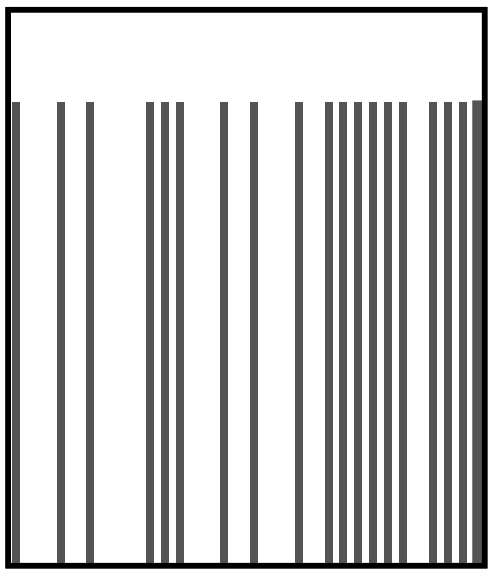

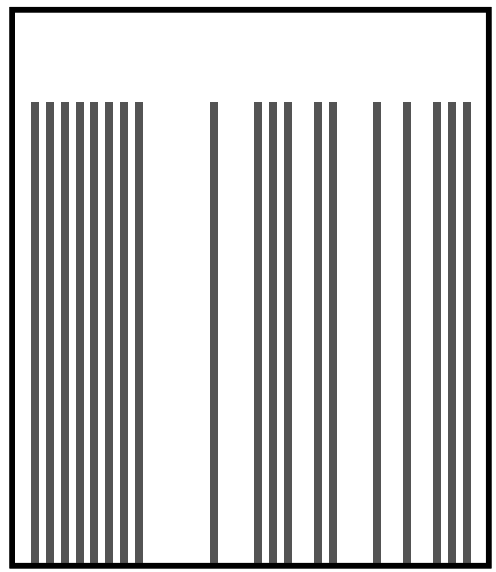

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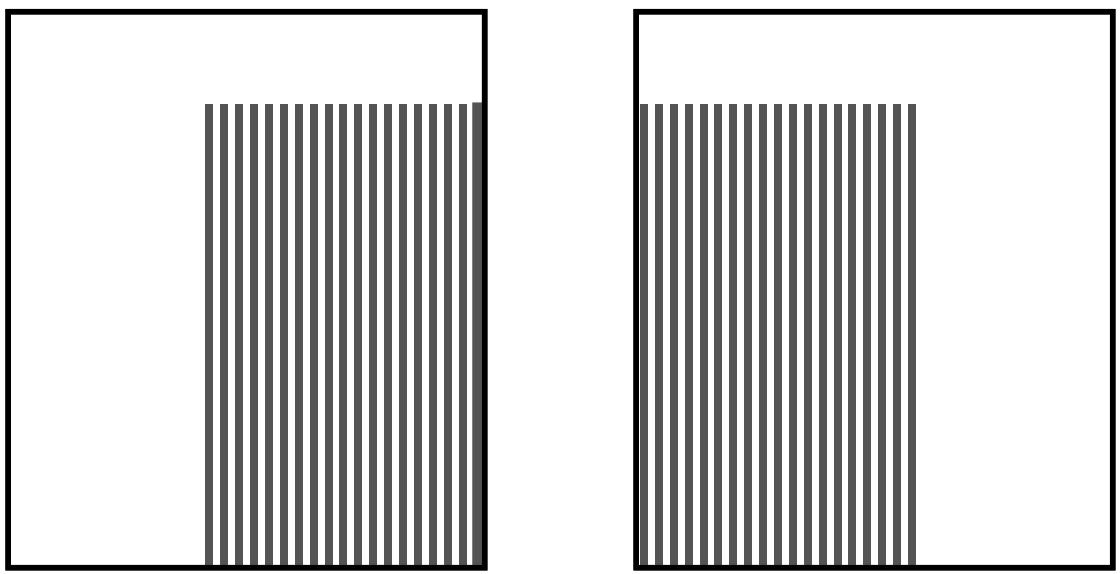

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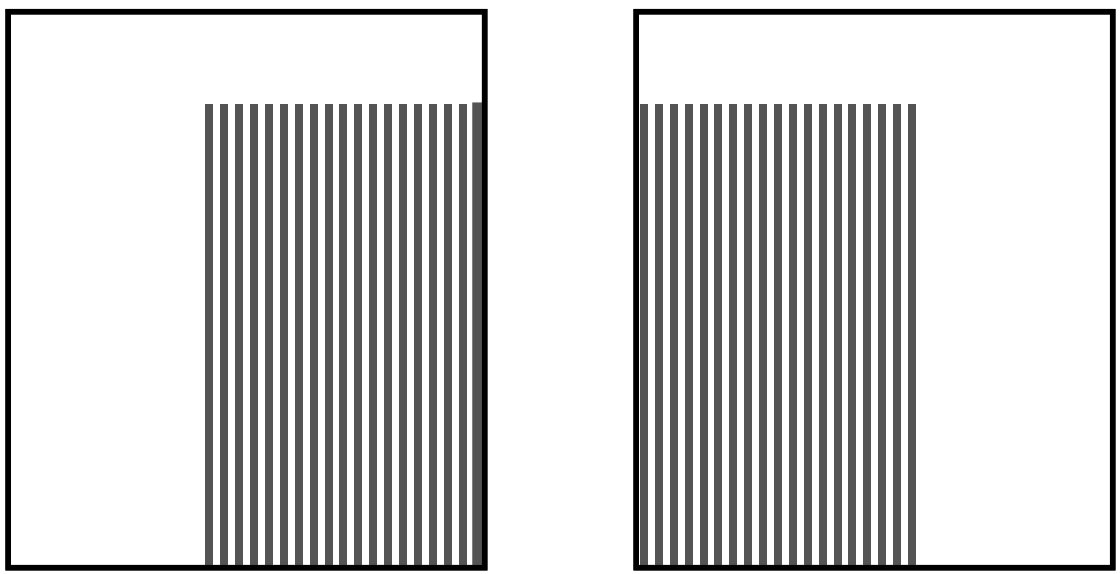

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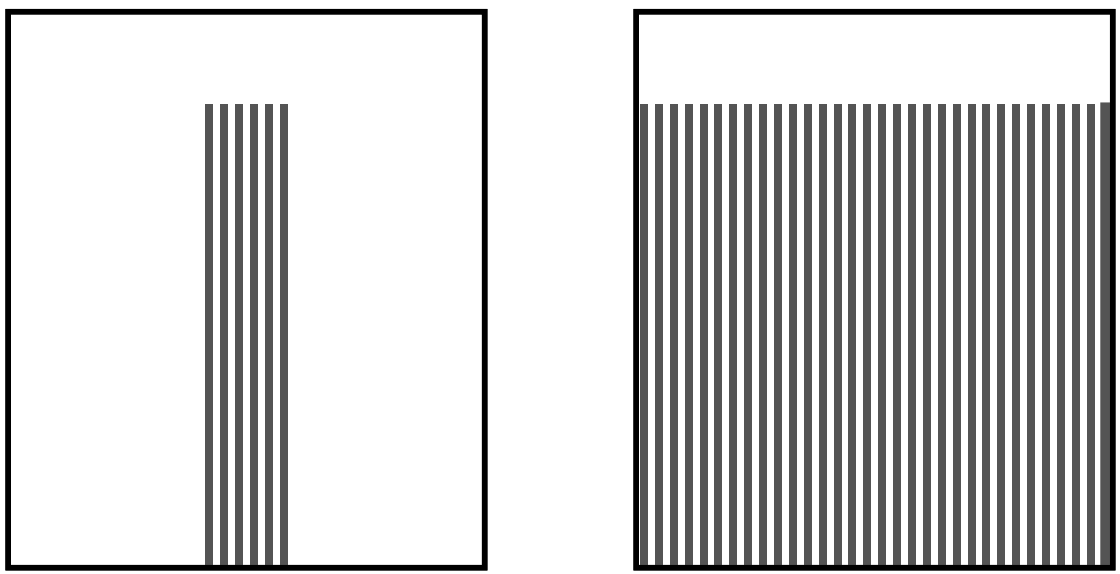

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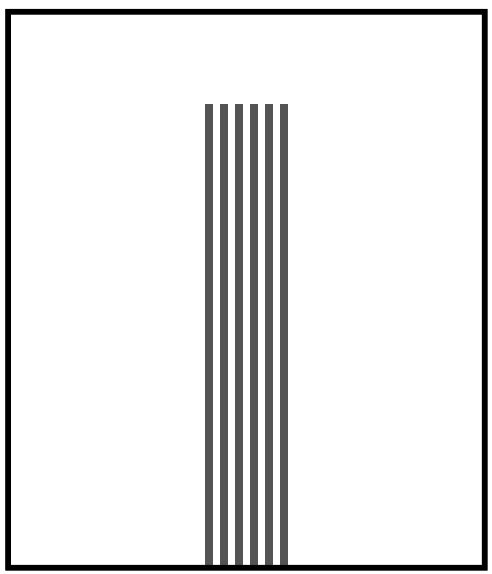

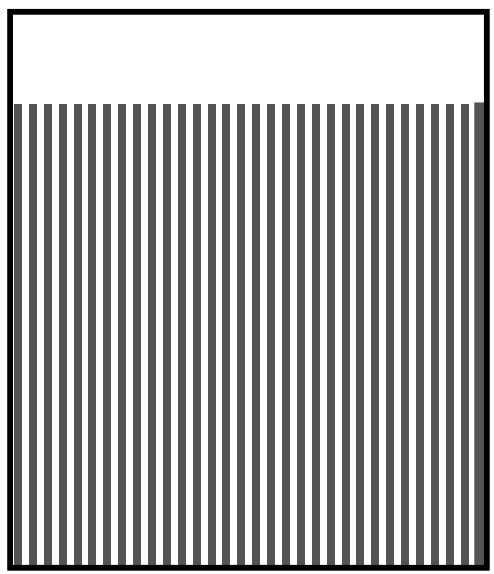

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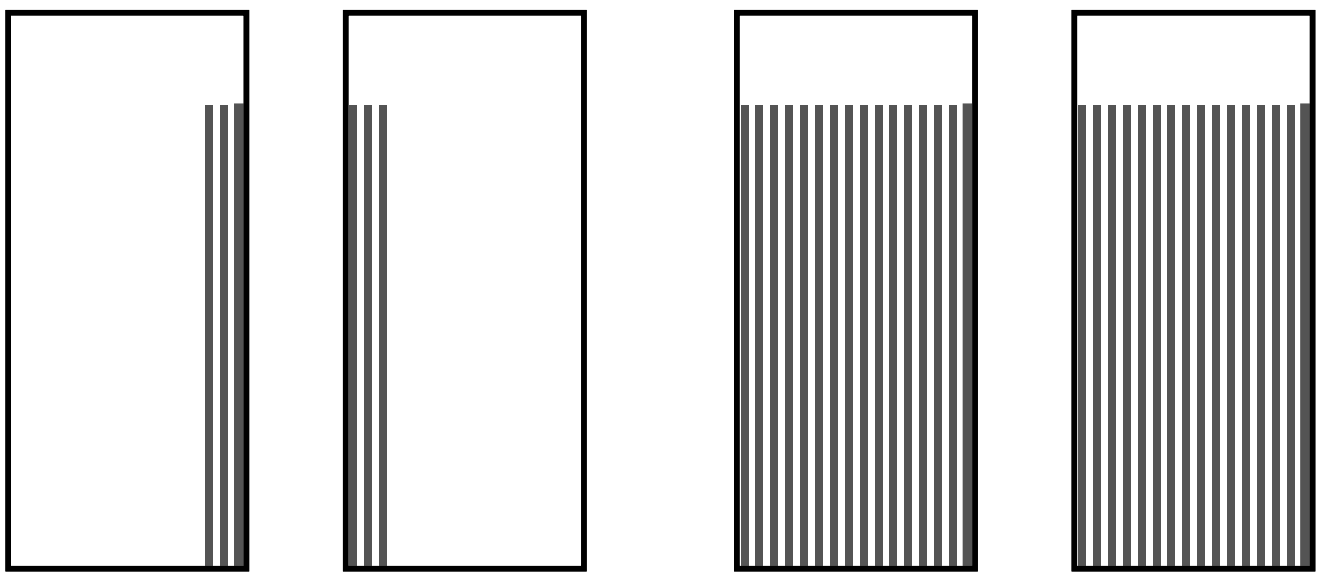

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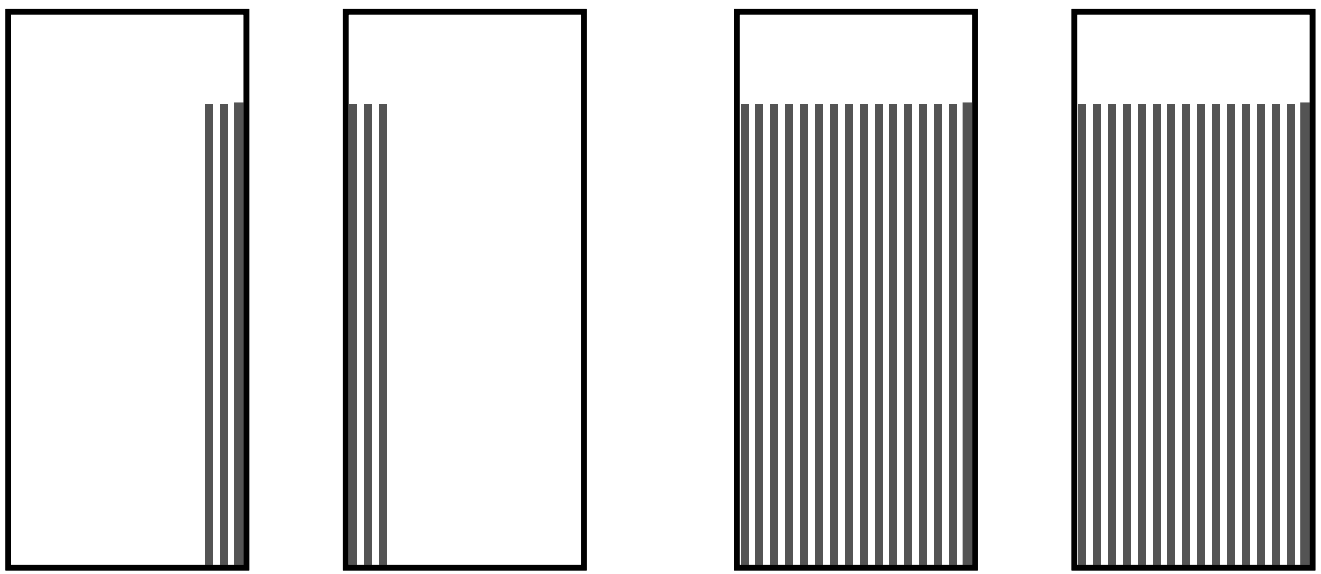

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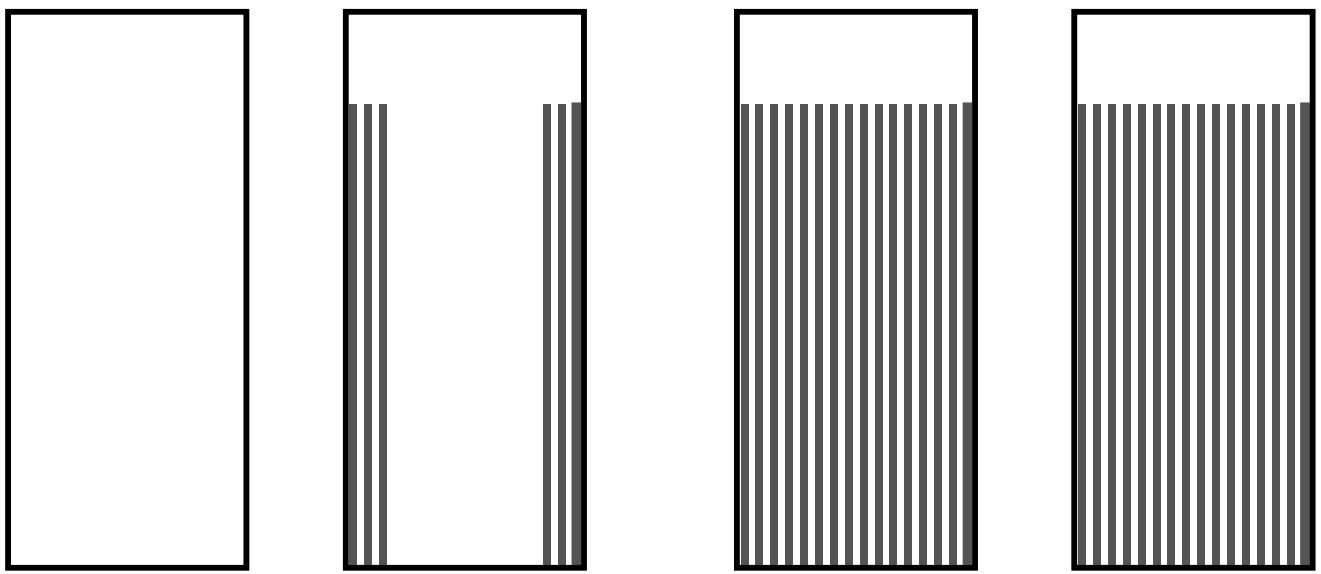

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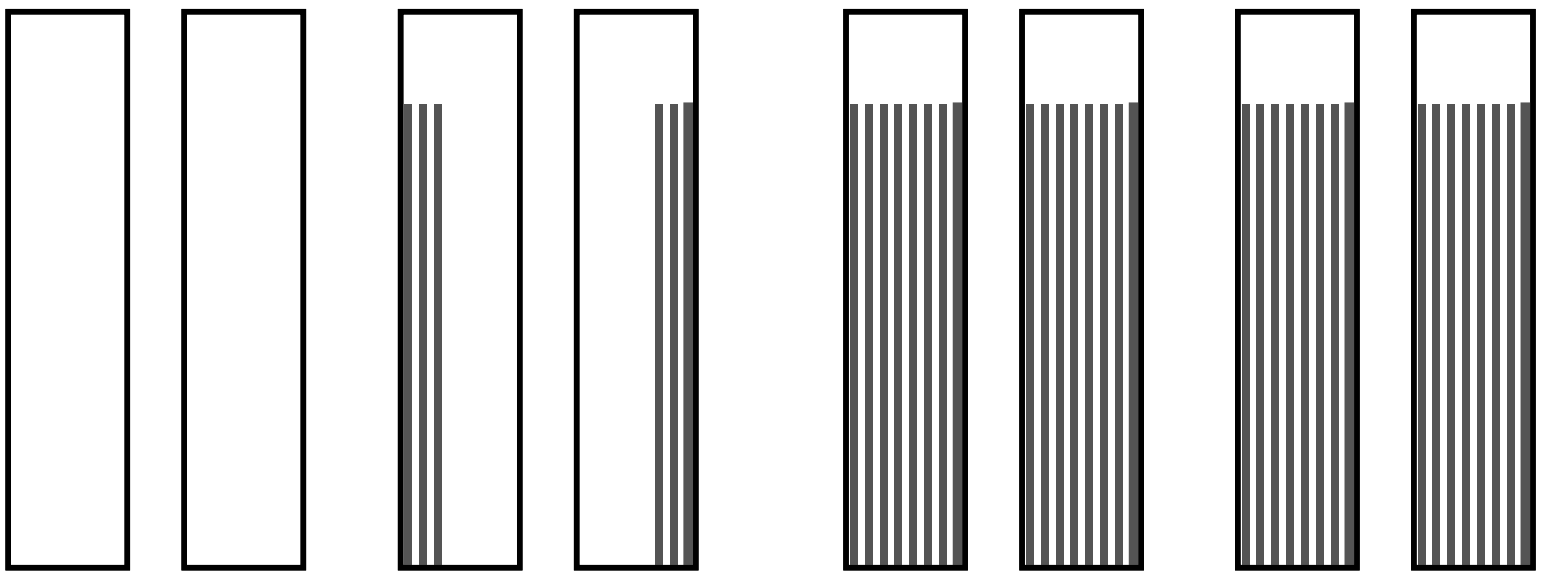

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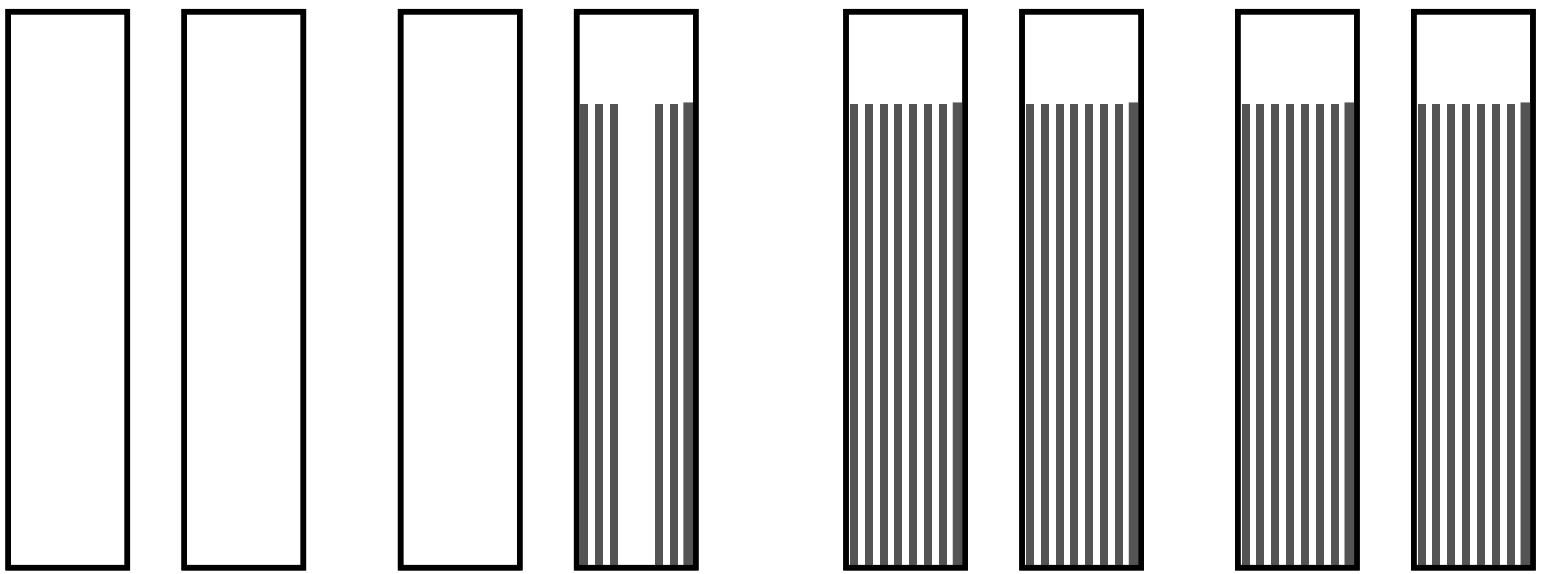

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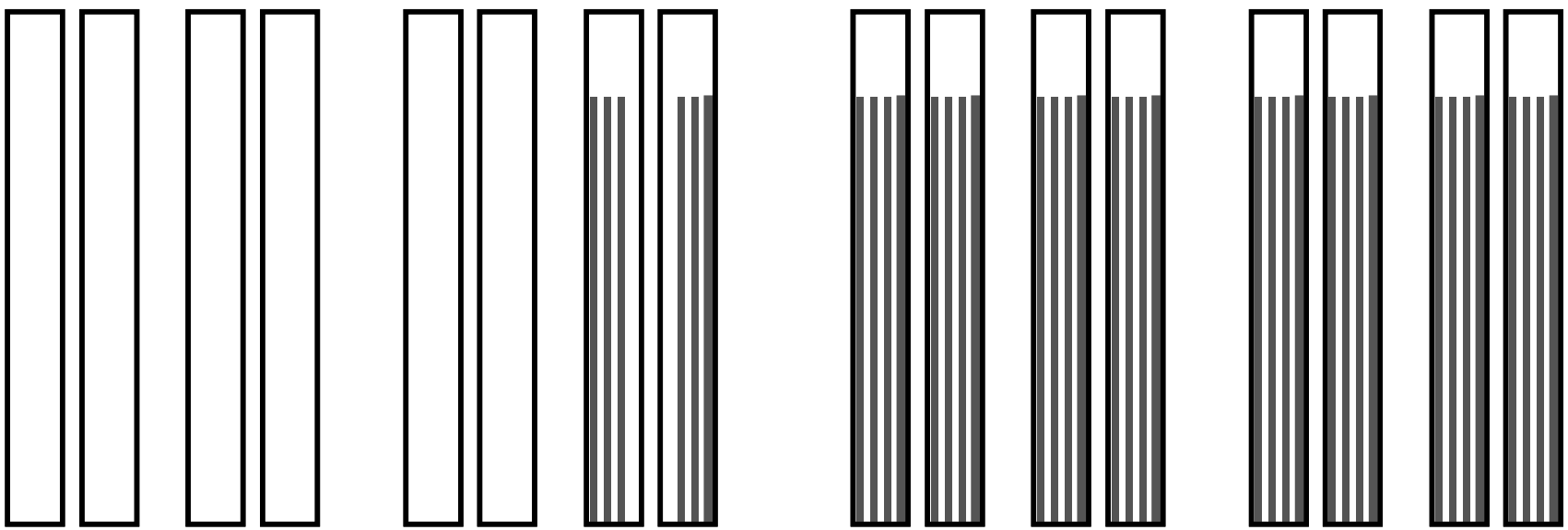

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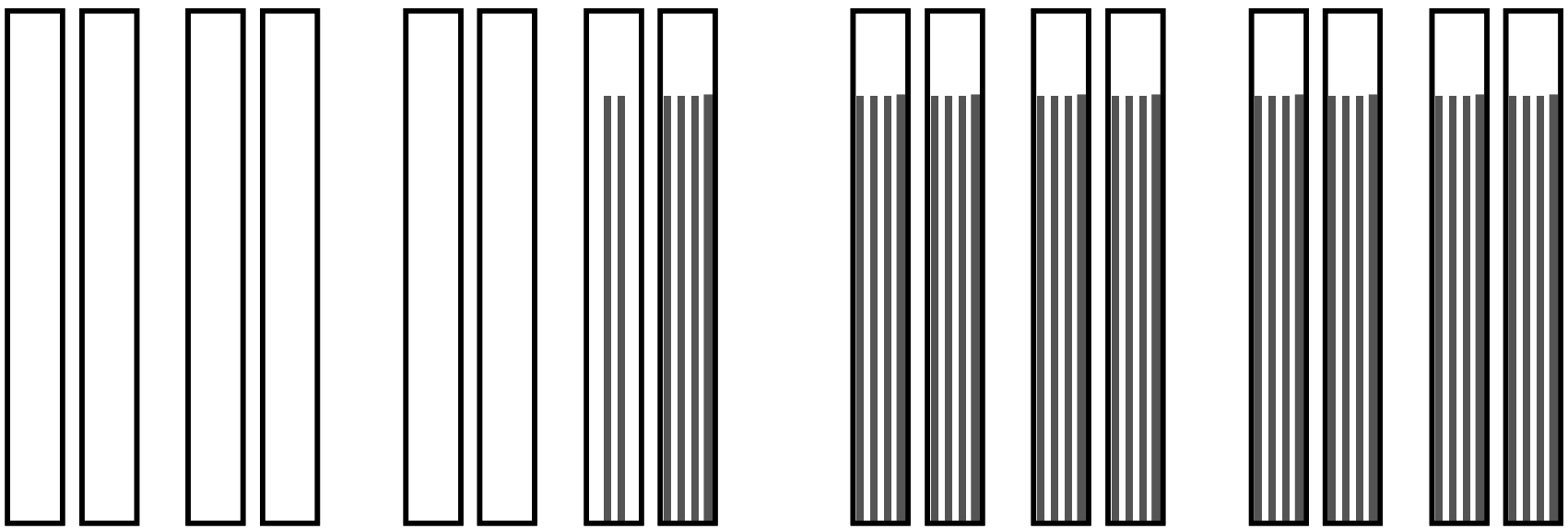

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Using the Free Theorems Generator

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 forall f::(t1,t1)->(t1,t1).
  forall g::(t2,t2)->(t2,t2).
   (forall (x,y) in lift_{(,)}(h,h).
     (f x,g y) in lift_{(,)}(h,h))
    ==> (forall xs::[t1].
          map h (sort f xs) = sort g (map h xs))

lift_{(,)}(h,h)
 = {((x1,x2),(y1,y2)) | (h x1 = y1)
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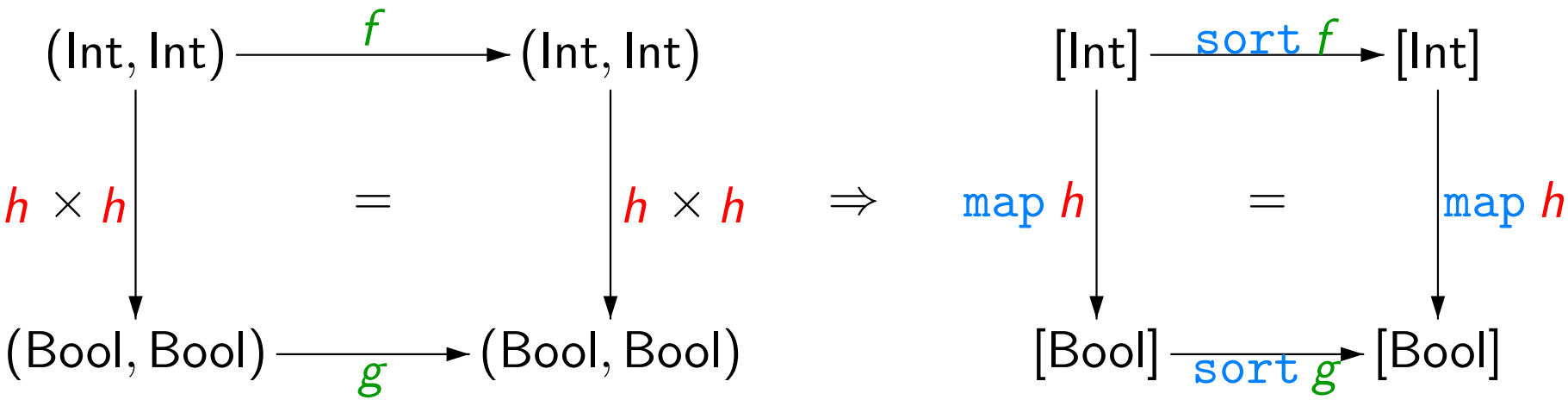
More Specific (and Intuitive)

For every `sort` :: $((\alpha, \alpha) \to (\alpha, \alpha)) \to [\alpha] \to [\alpha]$,
f :: $(\mathsf{Int}, \mathsf{Int}) \to (\mathsf{Int}, \mathsf{Int})$, g :: $(\mathsf{Bool}, \mathsf{Bool}) \to (\mathsf{Bool}, \mathsf{Bool})$, and
h :: $\mathsf{Int} \to \mathsf{Bool}$:



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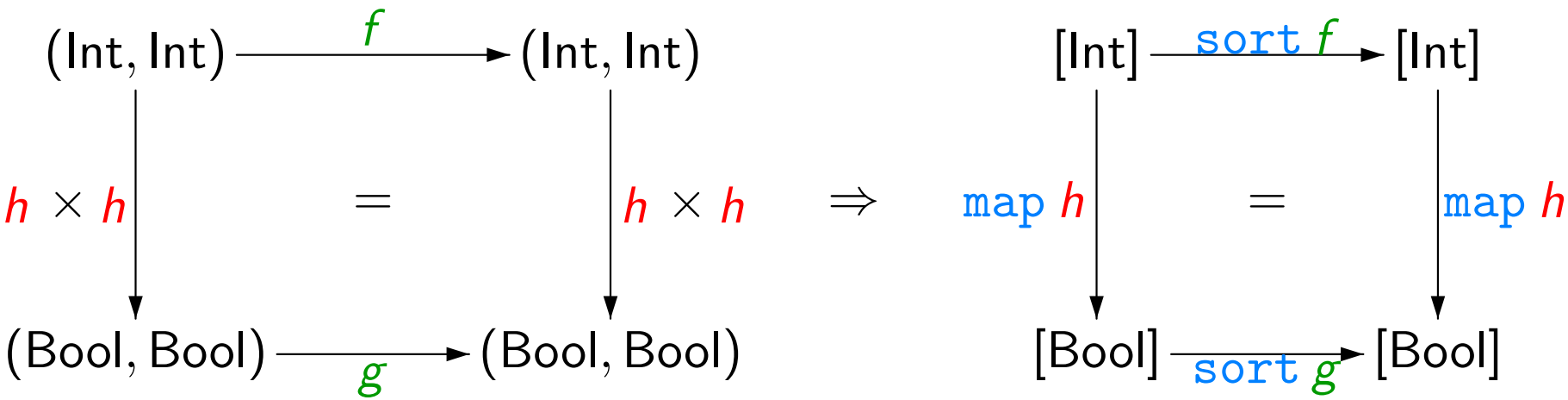

If f and g are as defined before, then the precondition is fulfilled for any h of the form $h\ x = n < x$ for some $n :: \mathsf{Int}$.

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- Good candidates: algorithms parametrised over some operation, like *cswap* $:: (\alpha, \alpha) \to (\alpha, \alpha)$ in the case of sorting.

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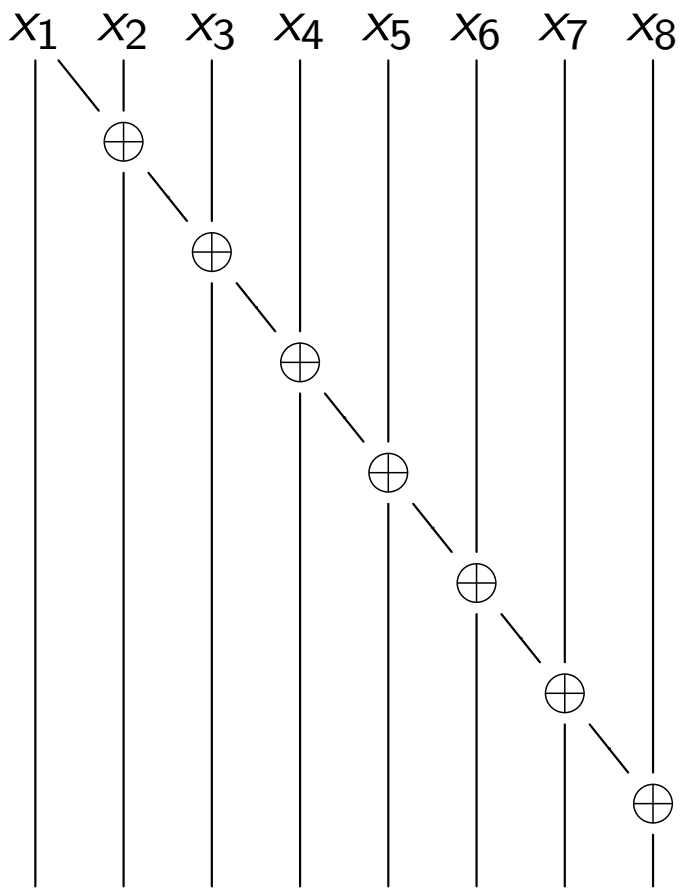

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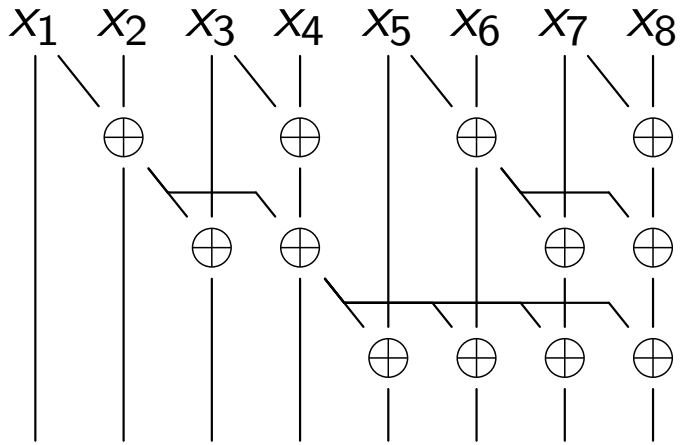

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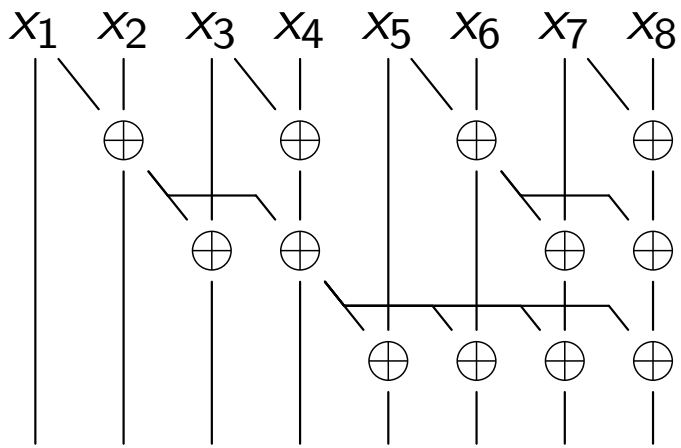
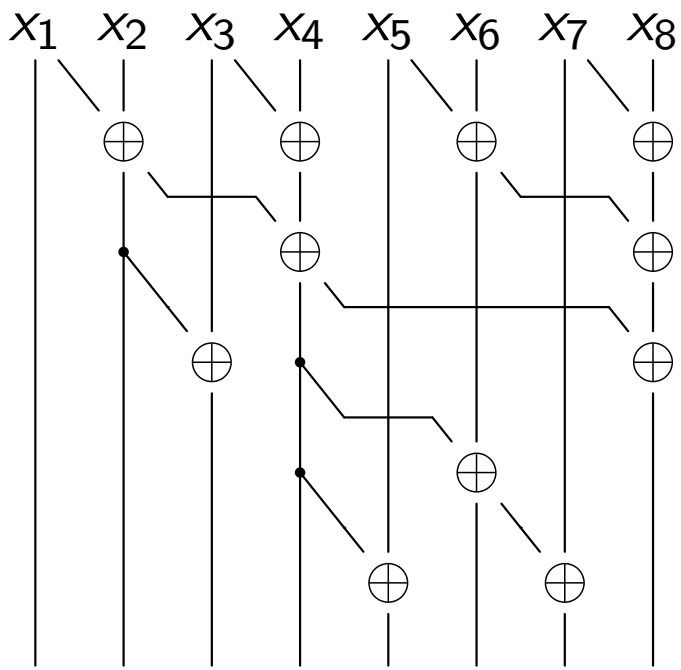


Or:



Or: ...

In Haskell

Functions of type:

$$\texttt{scanl1} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$$

In Haskell

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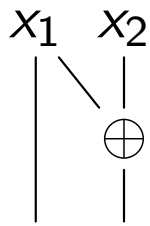
$$\texttt{scanl1} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$$

For example, à la [Sklansky 1960]:

```
sklansky :: (α → α → α) → [α] → [α]
sklansky (⊕) [x] = [x]
sklansky (⊕) xs  = us ++ vs
    where t       = ((length xs) + 1) `div` 2
          (ys, zs) = splitAt t xs
          us      = sklansky (⊕) ys
          vs      = [(last us) ⊕ v | v ← sklansky (⊕) zs]
```

Sklansky's Method Visualised



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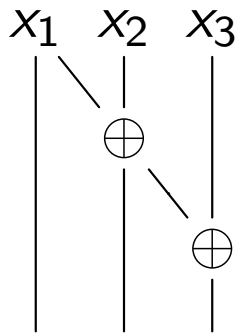
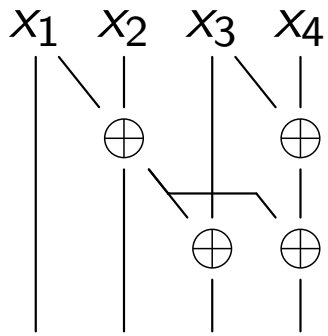


Sklansky's Method Visualised



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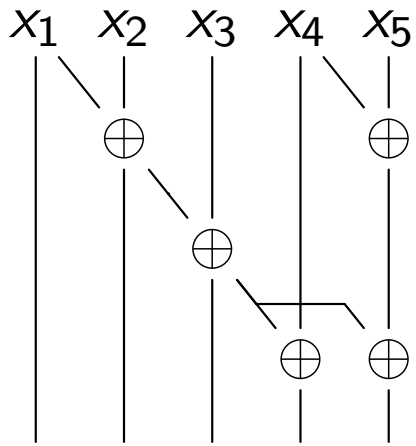

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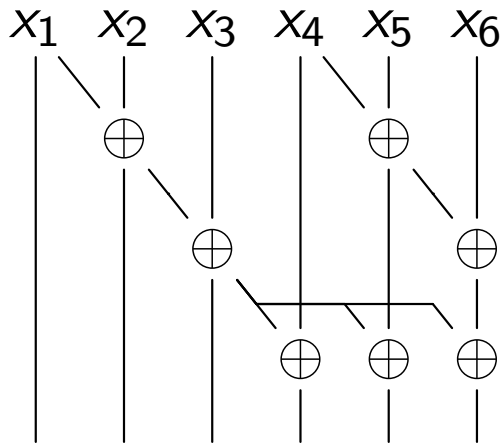

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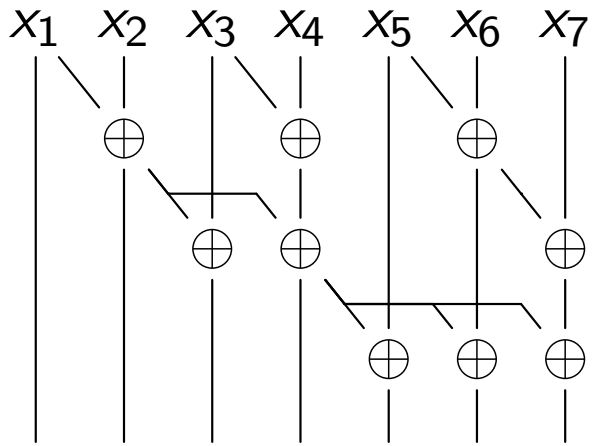

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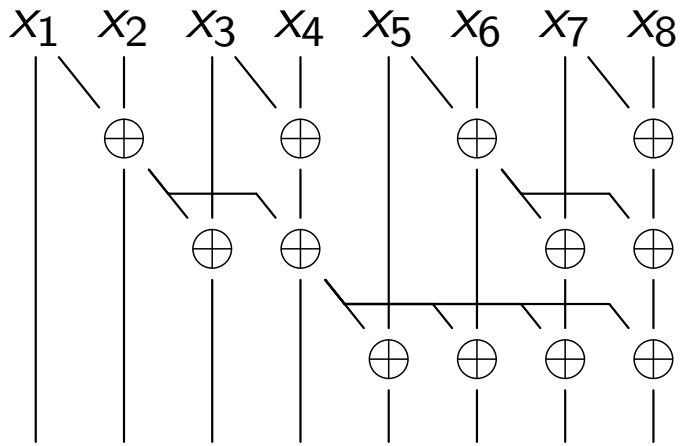

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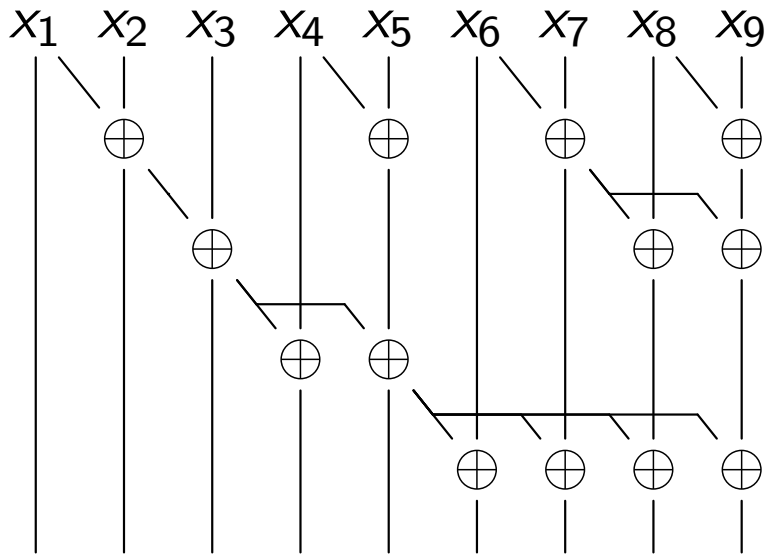

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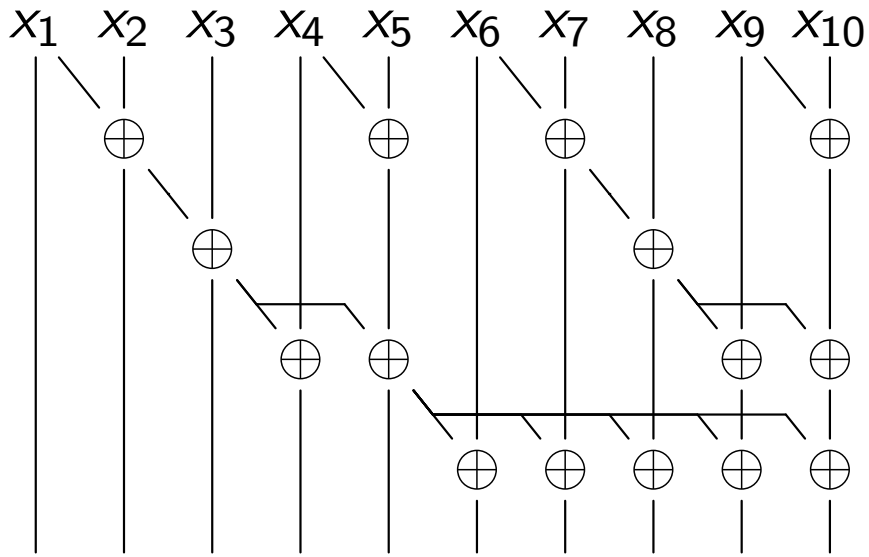

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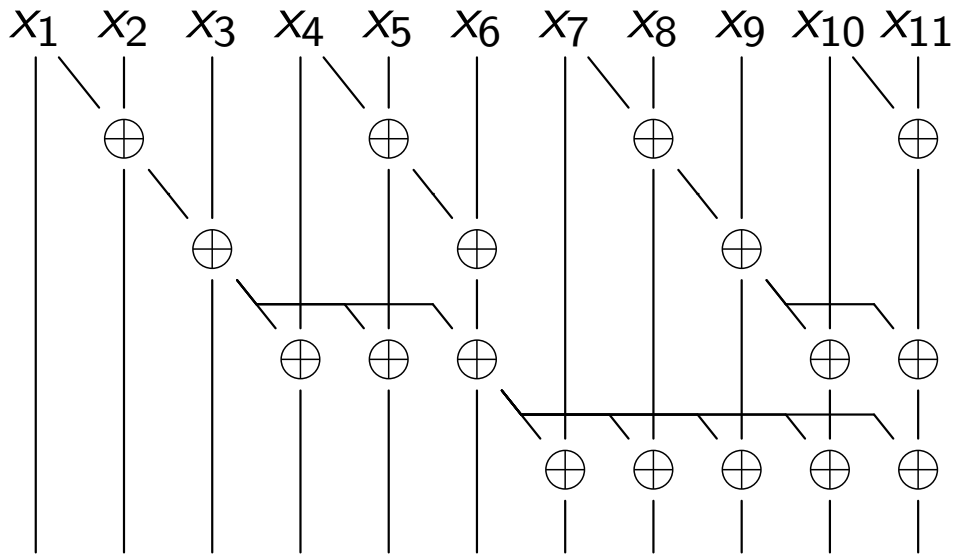

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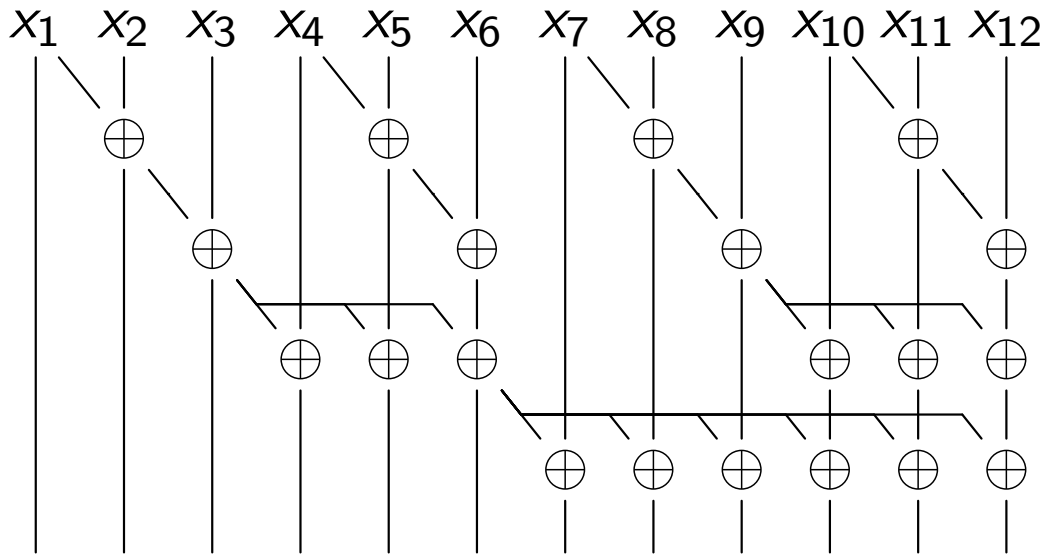

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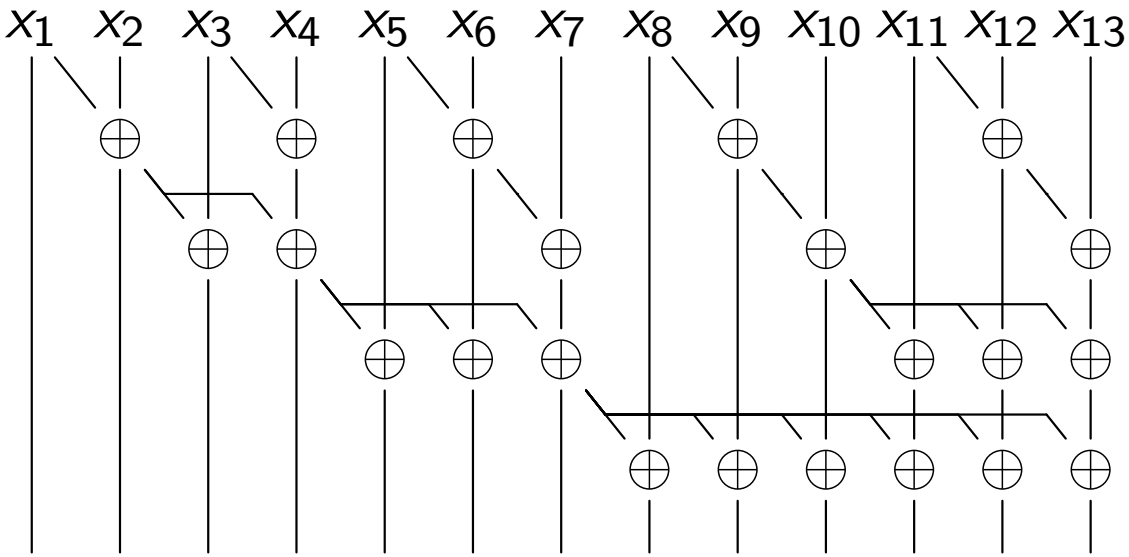

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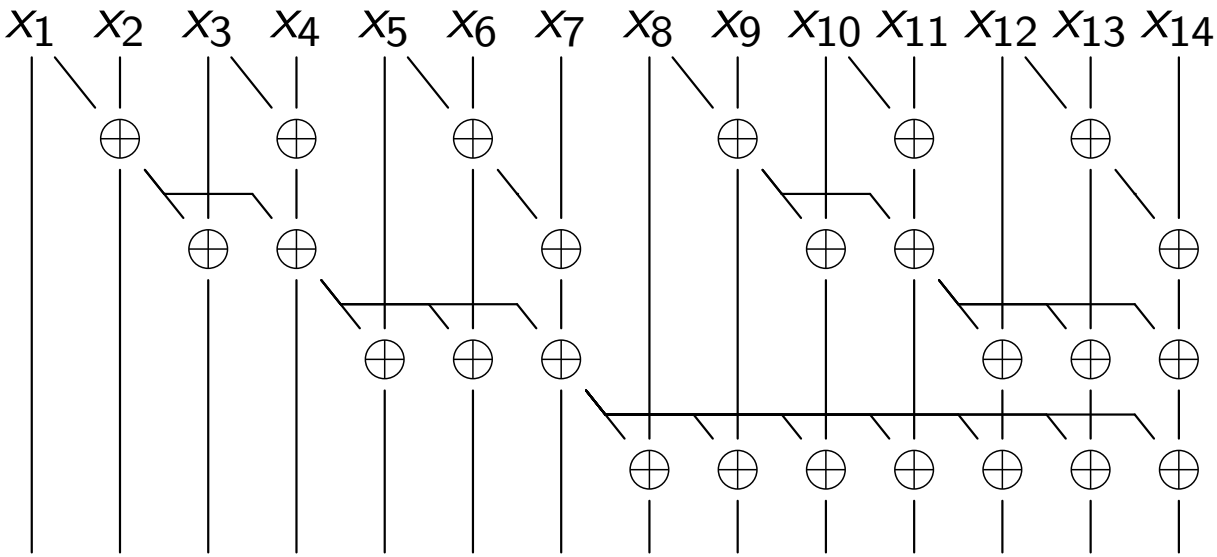

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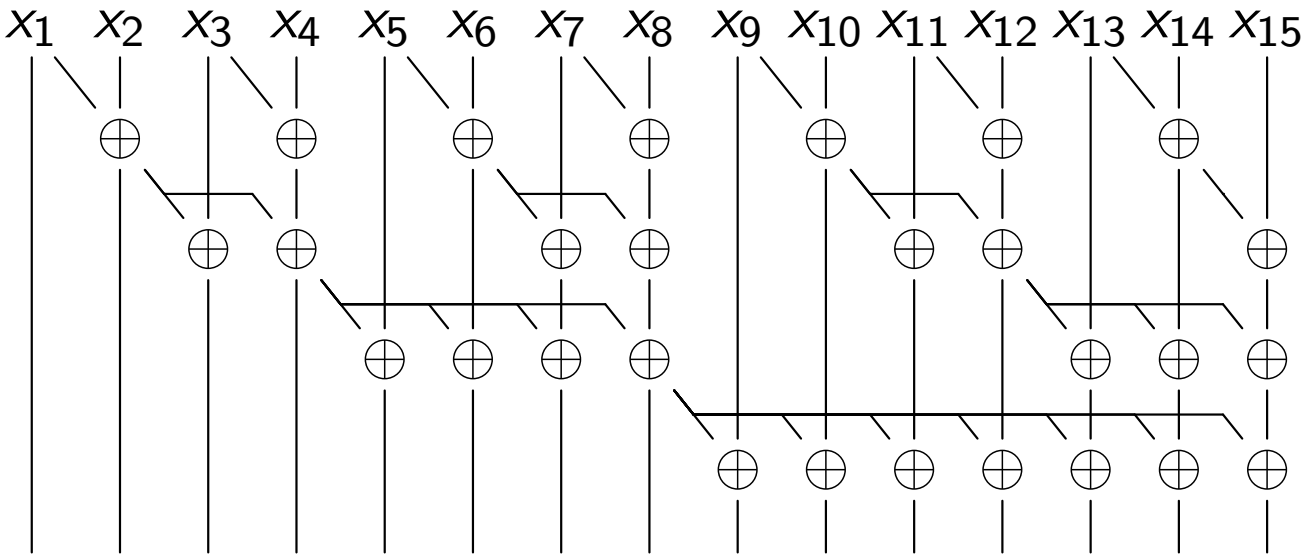

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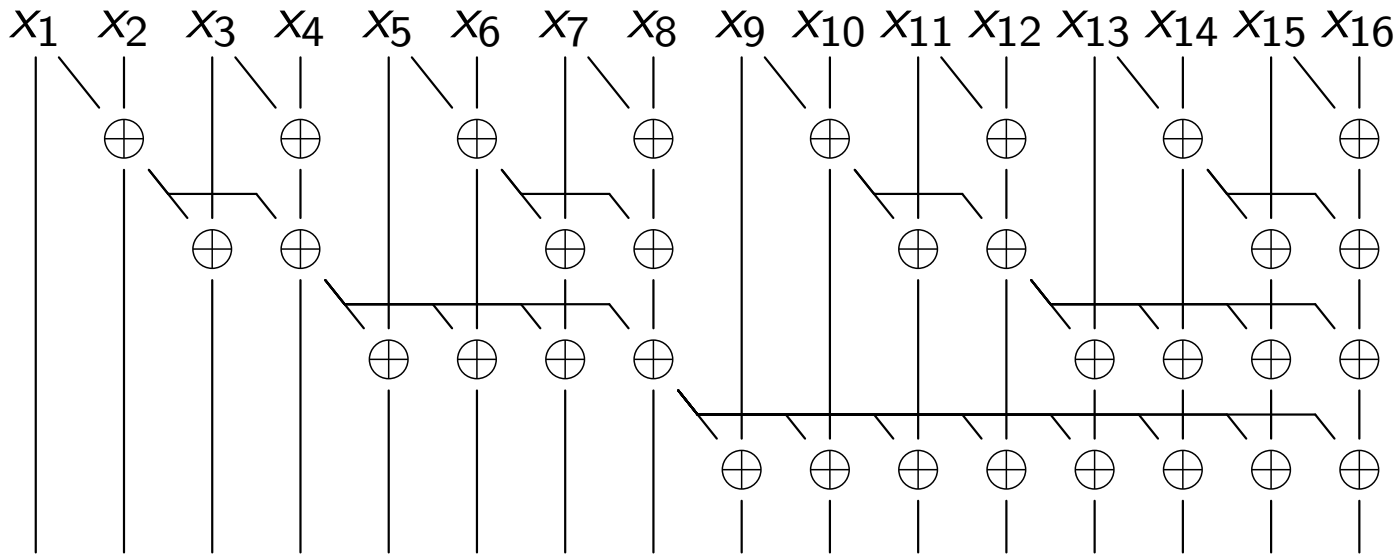

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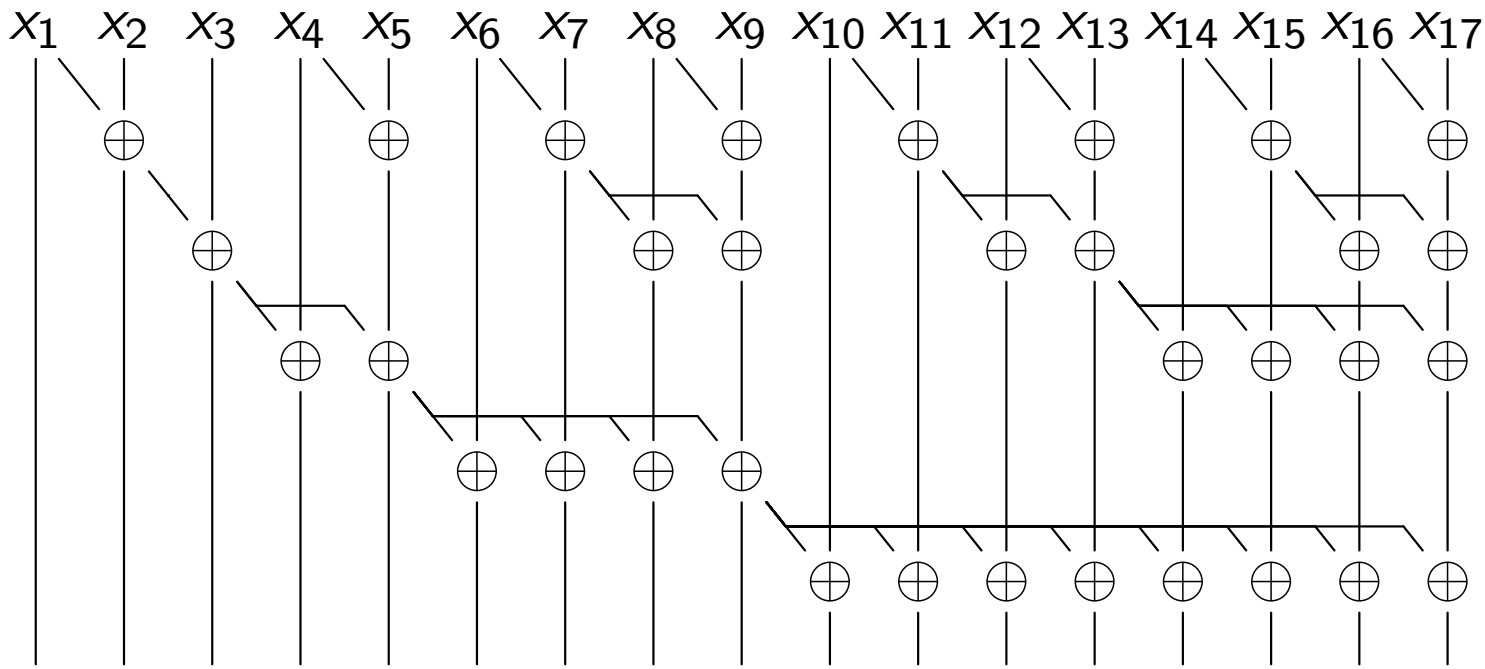

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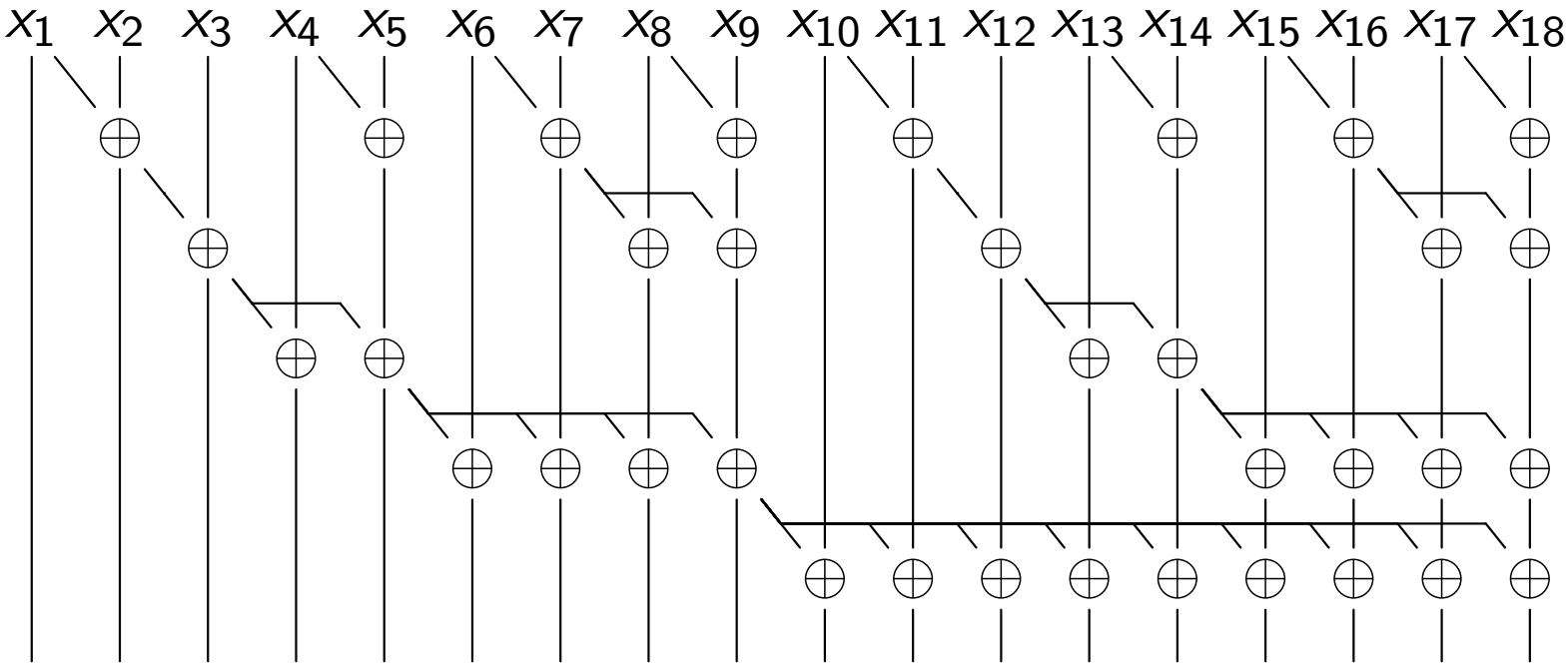

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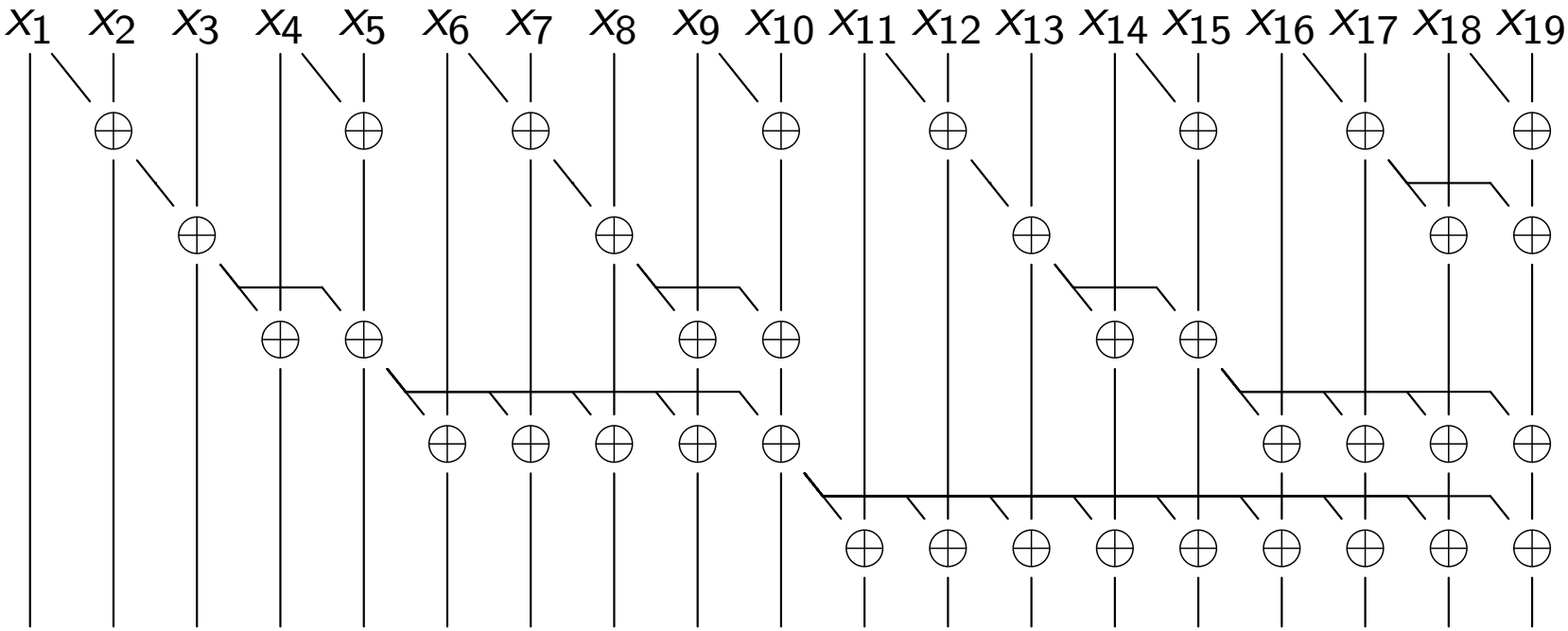
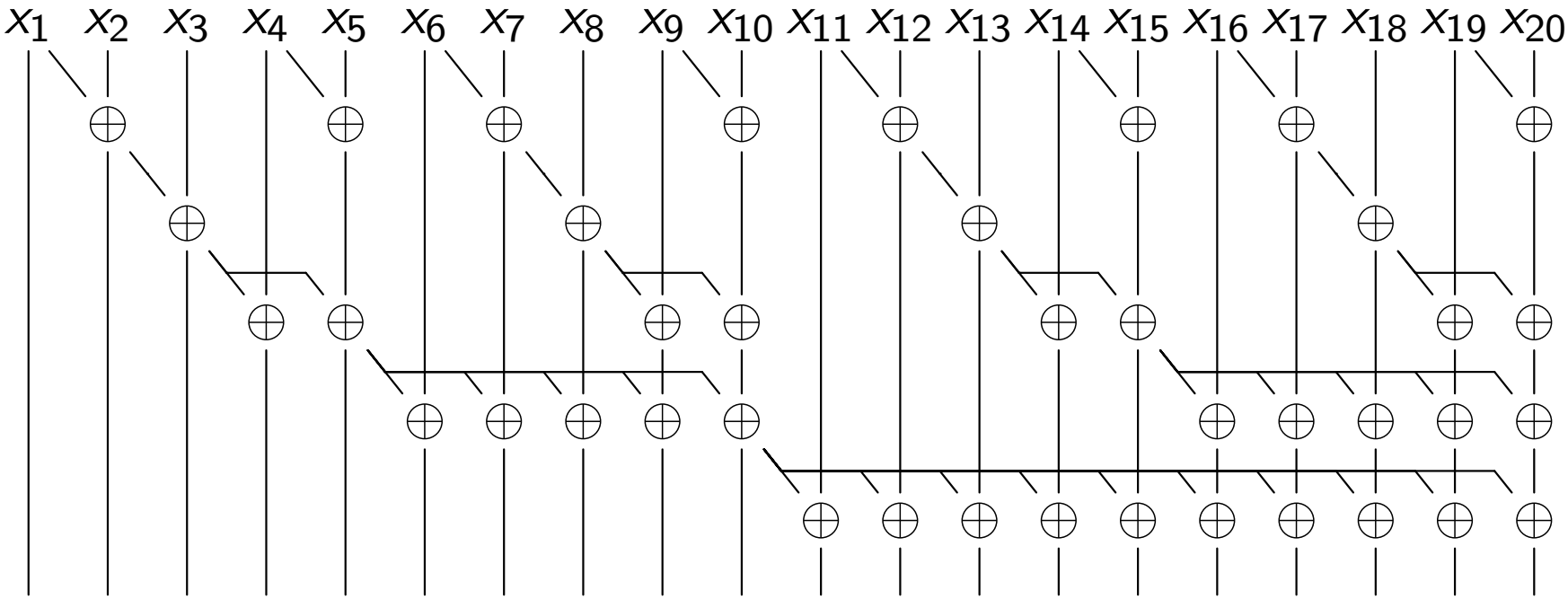


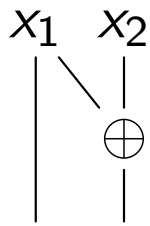
Sklansky's Method Visualised



Or, à la [Brent & Kung 1980] (code follows [Sheeran 2007])

$$
\begin{aligned}
&\texttt{brentKung} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha] \\
&\texttt{brentKung}\ (\oplus)\ [x] = [x] \\
&\texttt{brentKung}\ (\oplus)\ xs\ = \mathit{odds}\ (\mathit{riffle}\ (\mathit{par}\ (\mathit{unriffle}\ (\mathit{evens}\ xs)))) \\
&\quad \textbf{where}\ \mathit{evens}\ [\,] = [\,] \\
&\qquad\qquad \mathit{evens}\ [x] = [x] \\
&\qquad\qquad \mathit{evens}\ (x : y : zs) = [x, x \oplus y] +\!\!+ \mathit{evens}\ zs \\
&\qquad\qquad \mathit{unriffle}\ [\,] = ([\,], [\,]) \\
&\qquad\qquad \mathit{unriffle}\ [x] = ([x], [\,]) \\
&\qquad\qquad \mathit{unriffle}\ (x : y : zs) = (x : xs, y : ys) \\
&\qquad\qquad\quad \textbf{where}\ (xs, ys) = \mathit{unriffle}\ zs \\
&\qquad\qquad \mathit{par}\ (xs, ys) = (xs, \texttt{brentKung}\ (\oplus)\ ys) \\
&\qquad\qquad \mathit{riffle}\ ([\,], [\,]) = [\,] \\
&\qquad\qquad \mathit{riffle}\ ([x], [\,]) = [x] \\
&\qquad\qquad \mathit{riffle}\ (x : xs, y : ys) = x : y : \mathit{riffle}\ (xs, ys) \\
&\qquad\qquad \mathit{odds}\ (x : xs) = x : \mathit{evens}\ xs
\end{aligned}
$$

Brent & Kung's Method Visualised



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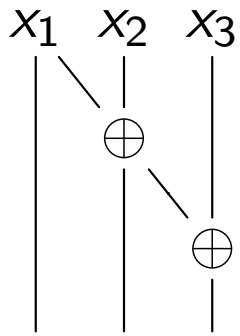

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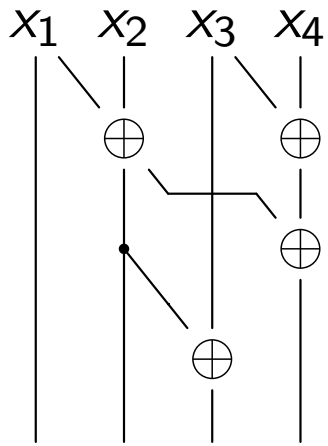
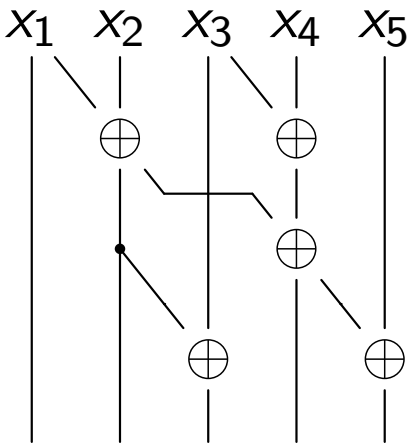


Brent & Kung's Method Visualised



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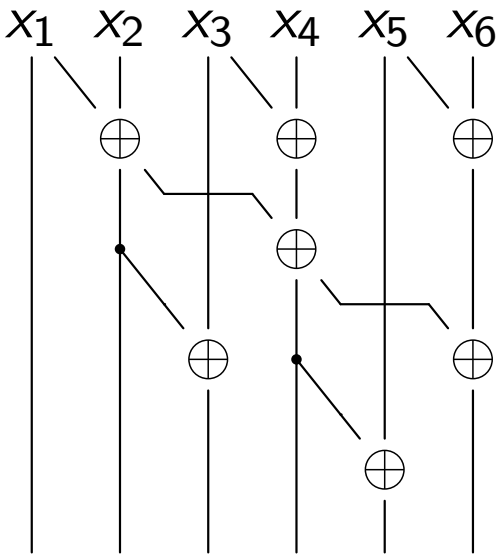
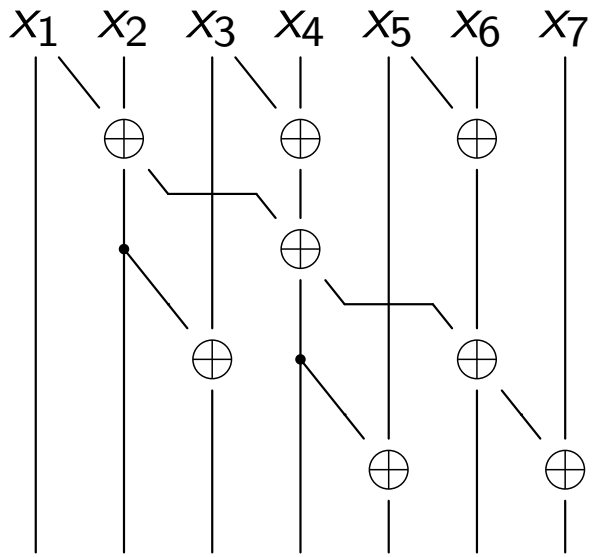


Brent & Kung's Method Visualised



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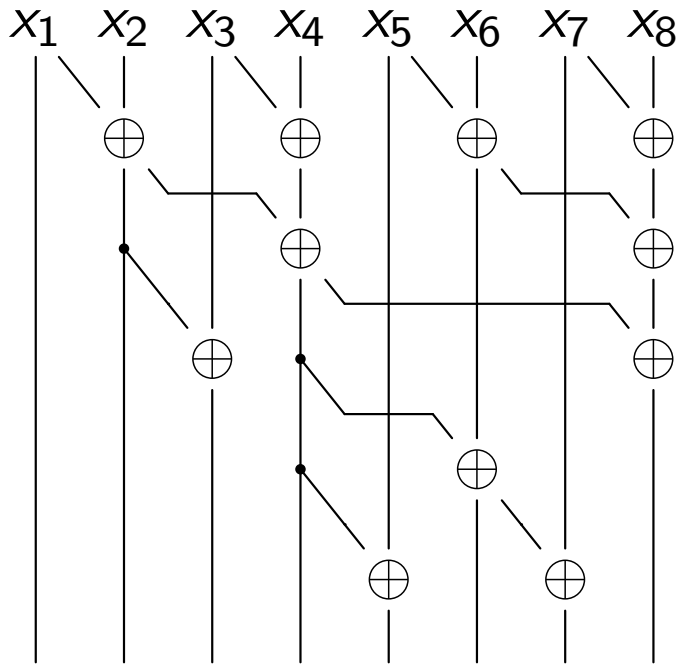

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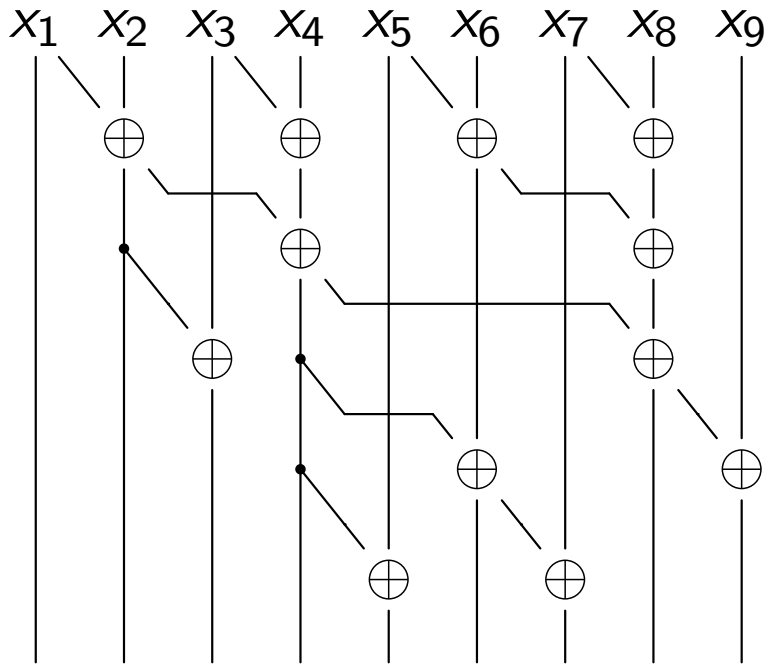

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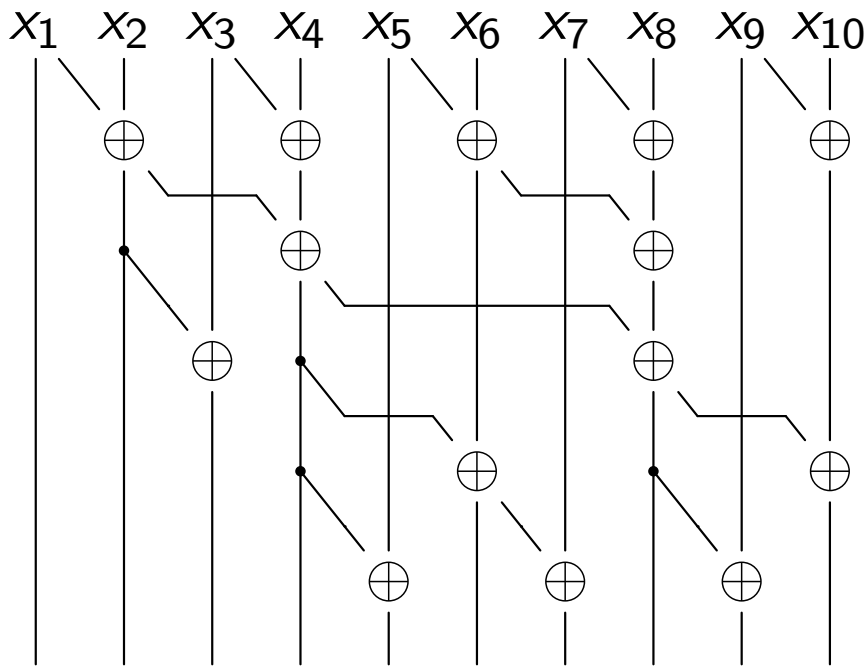

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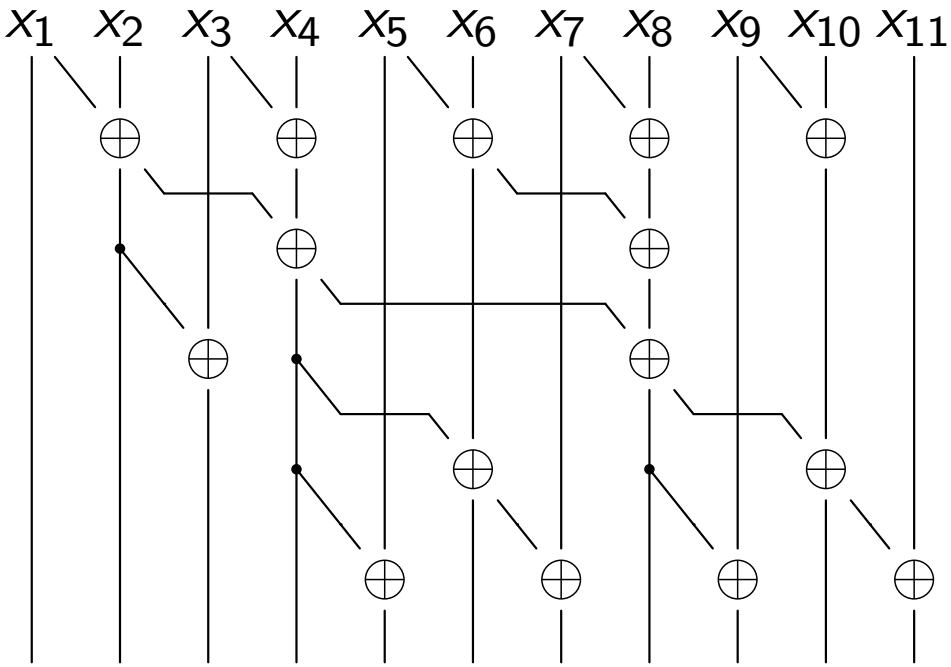

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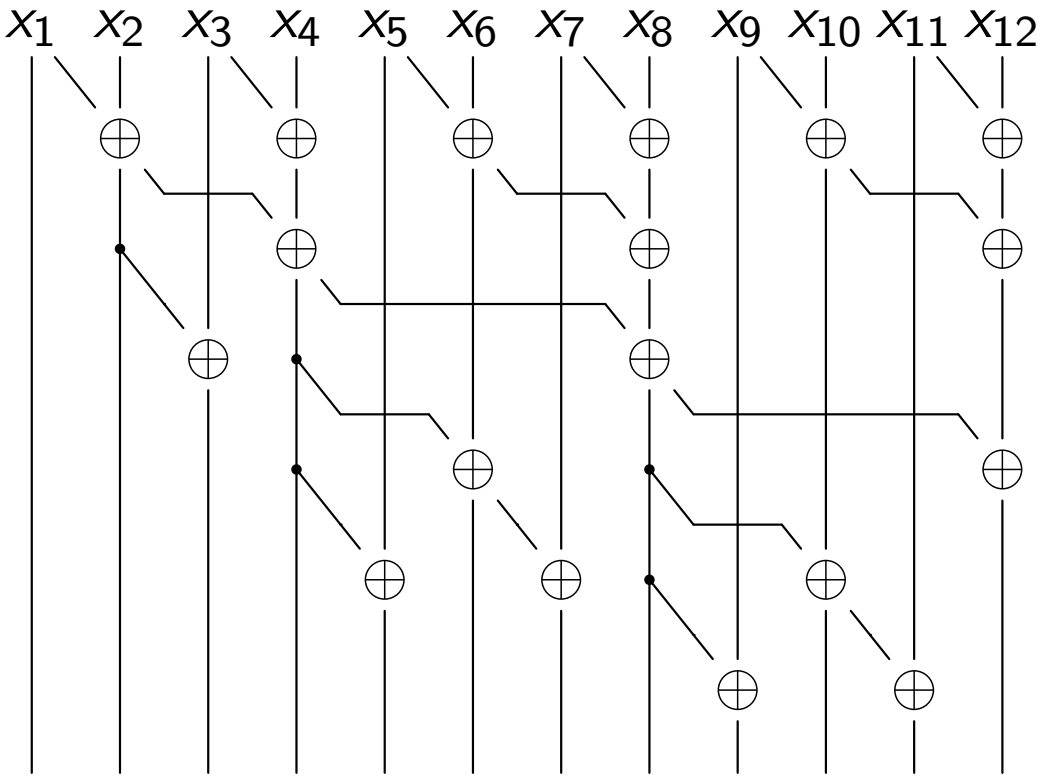

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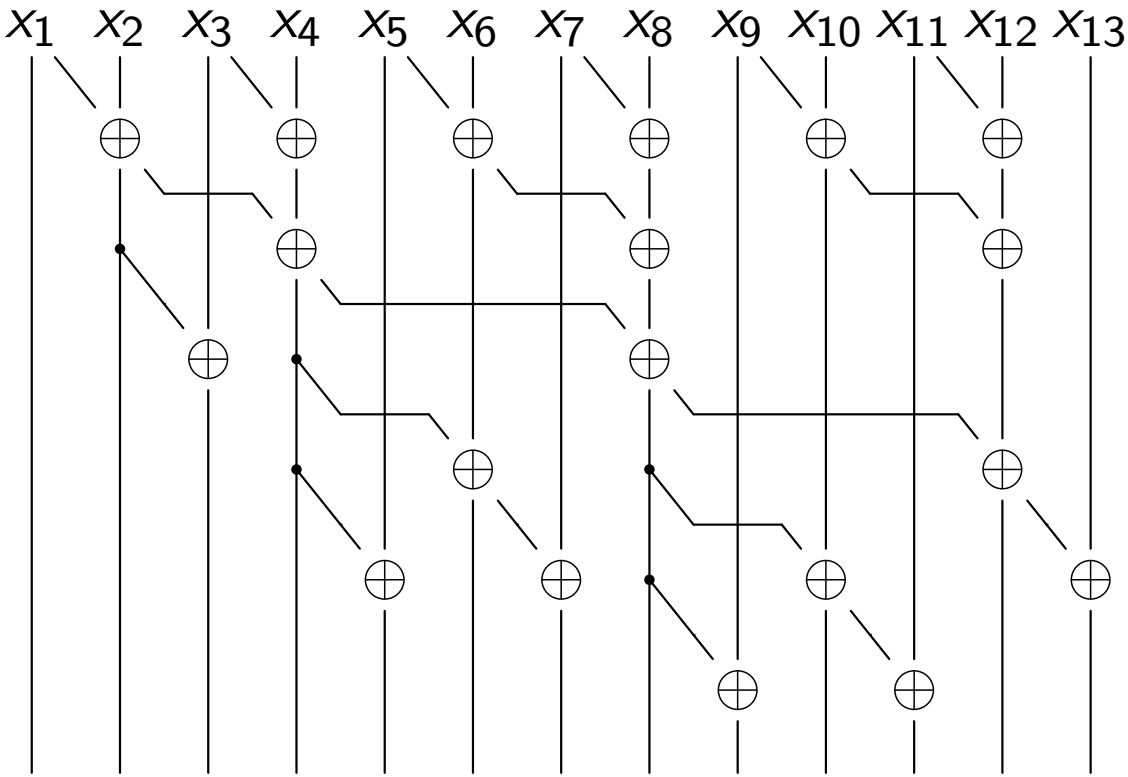

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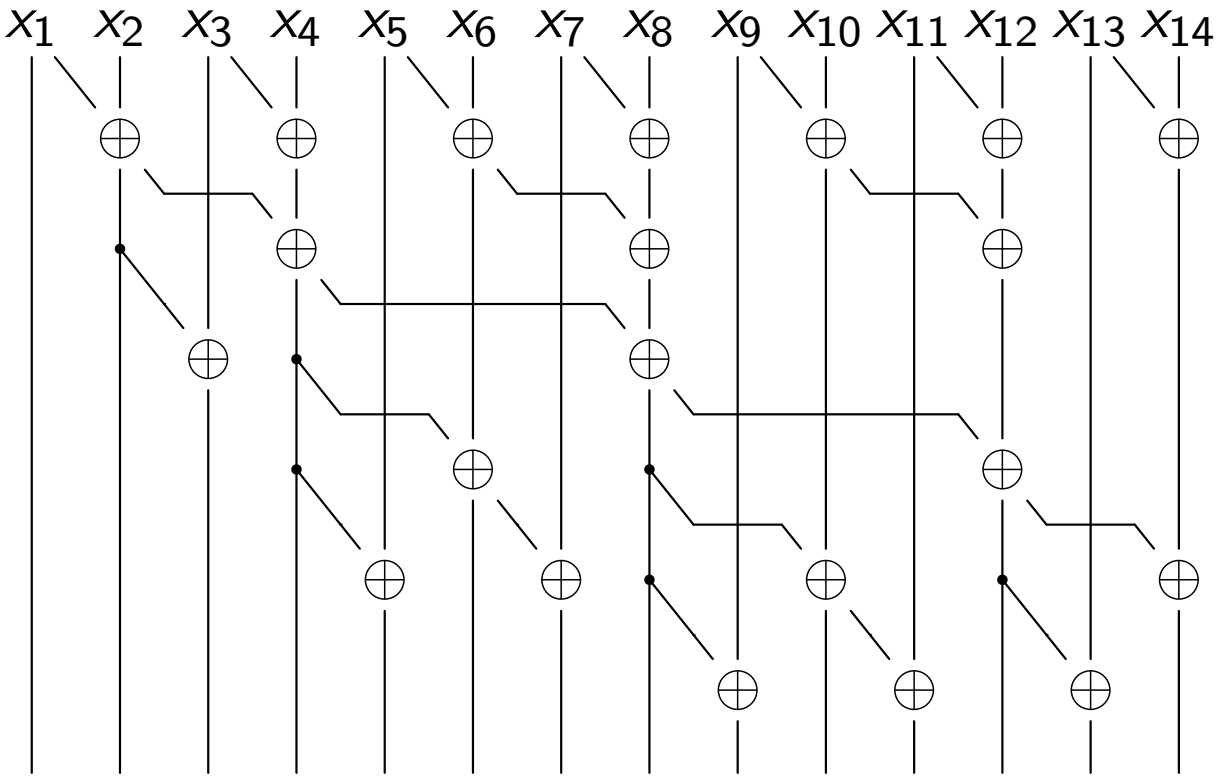

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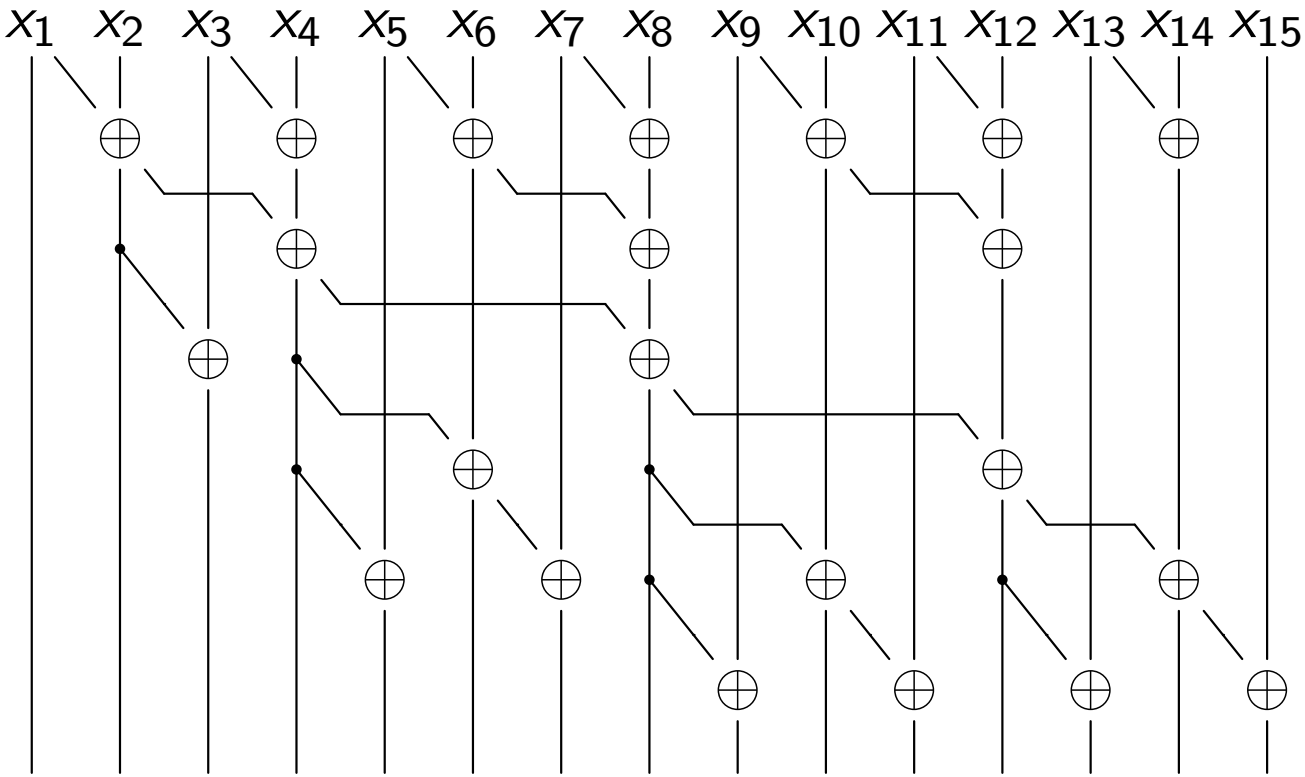

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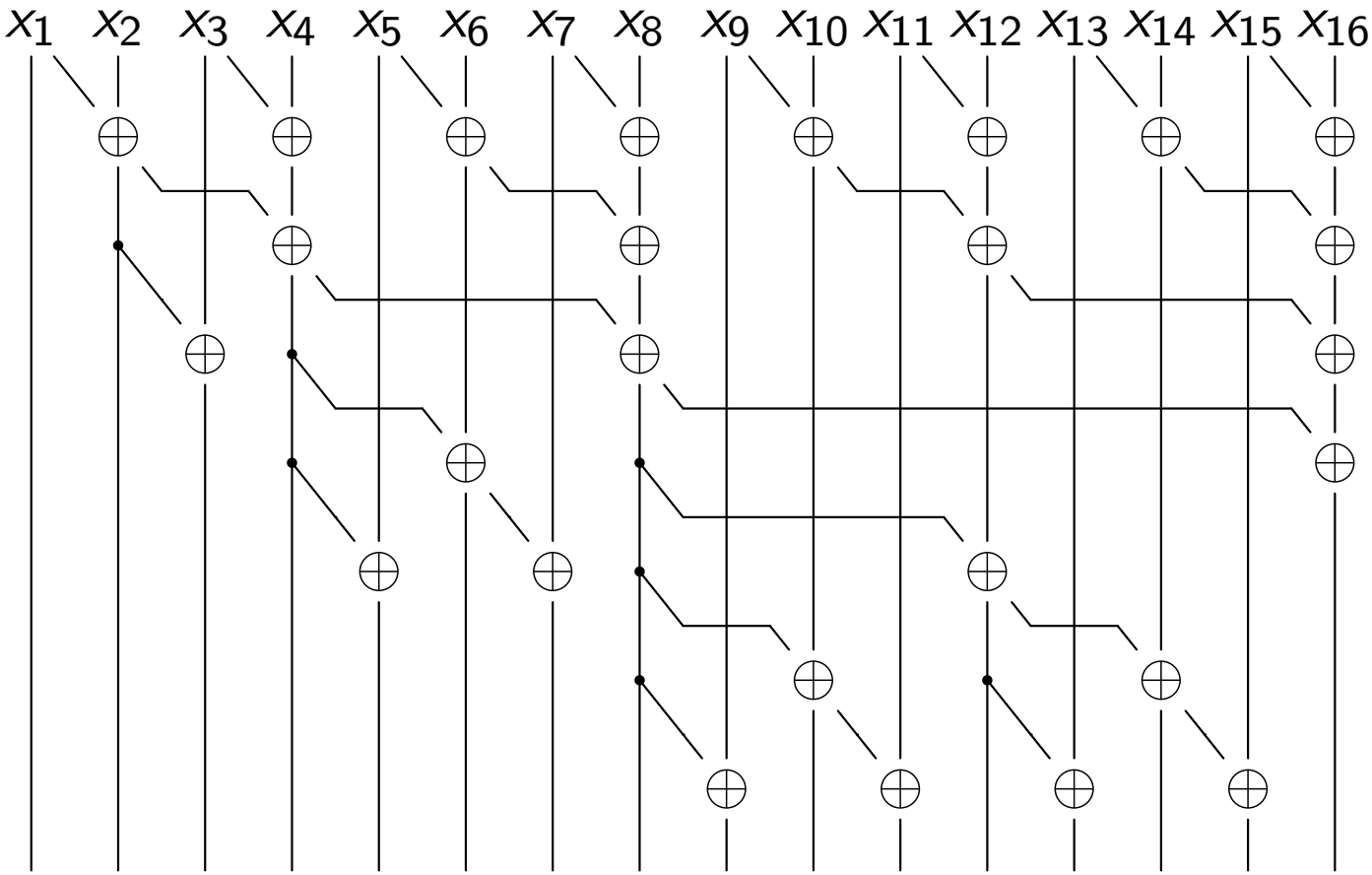

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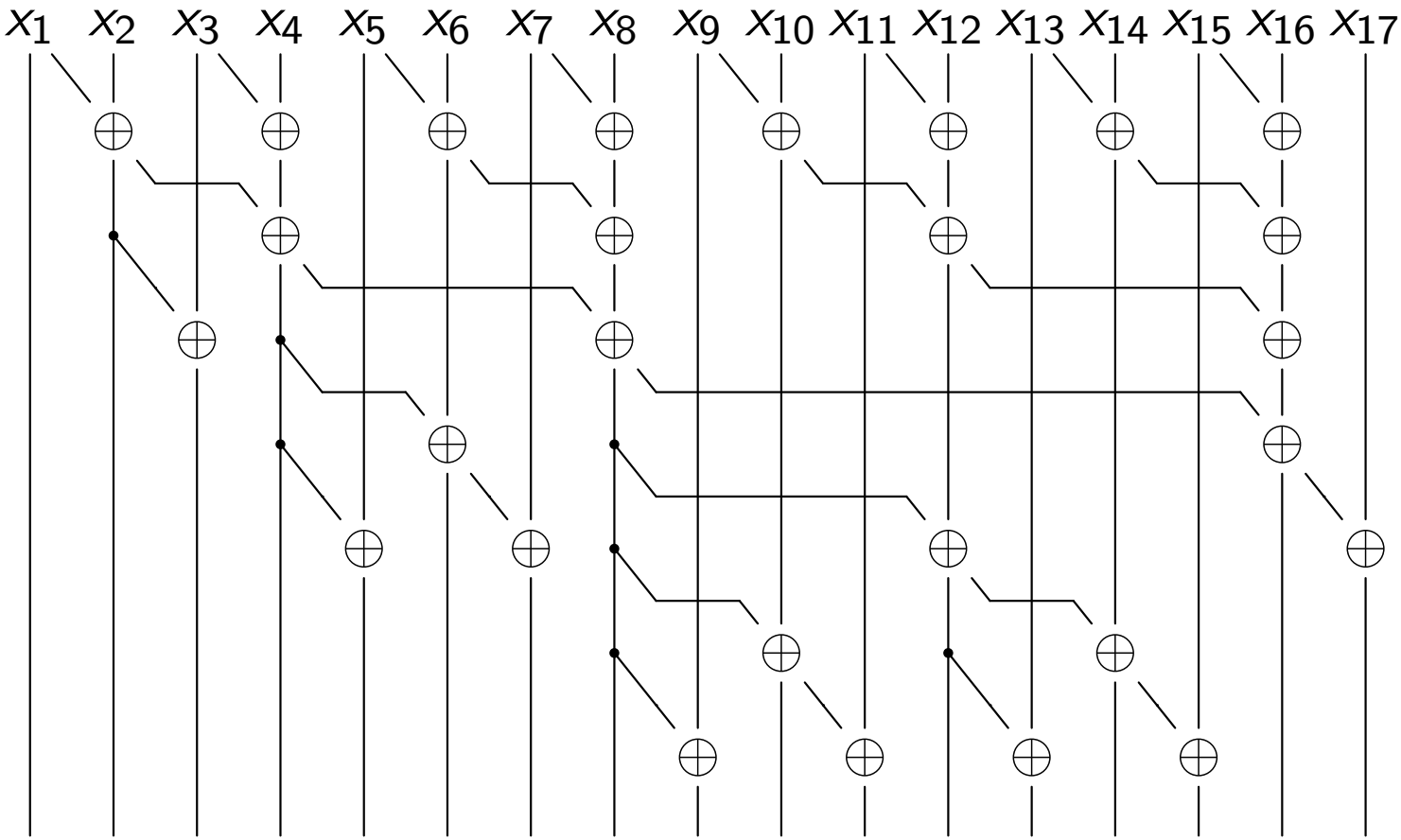

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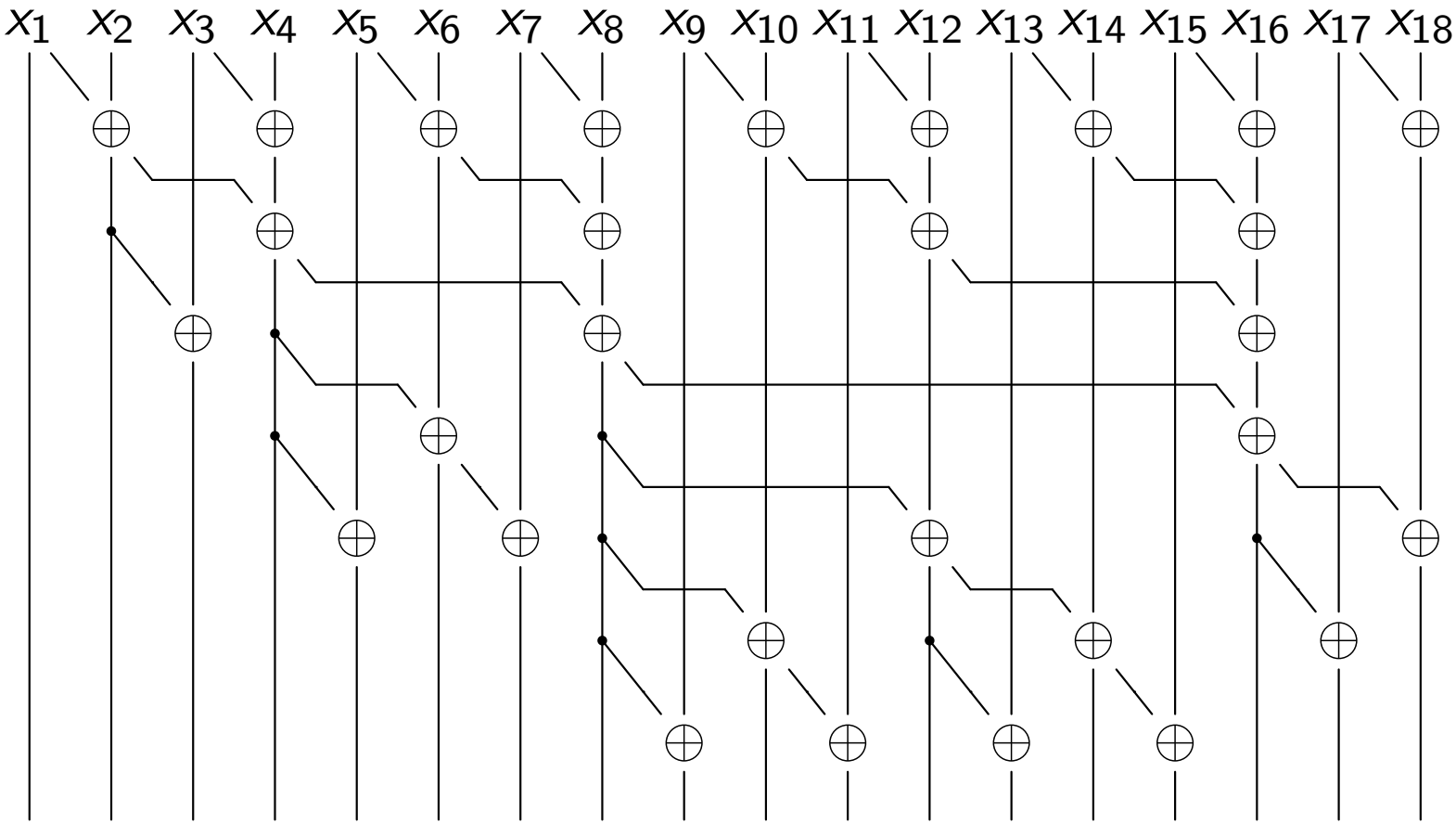

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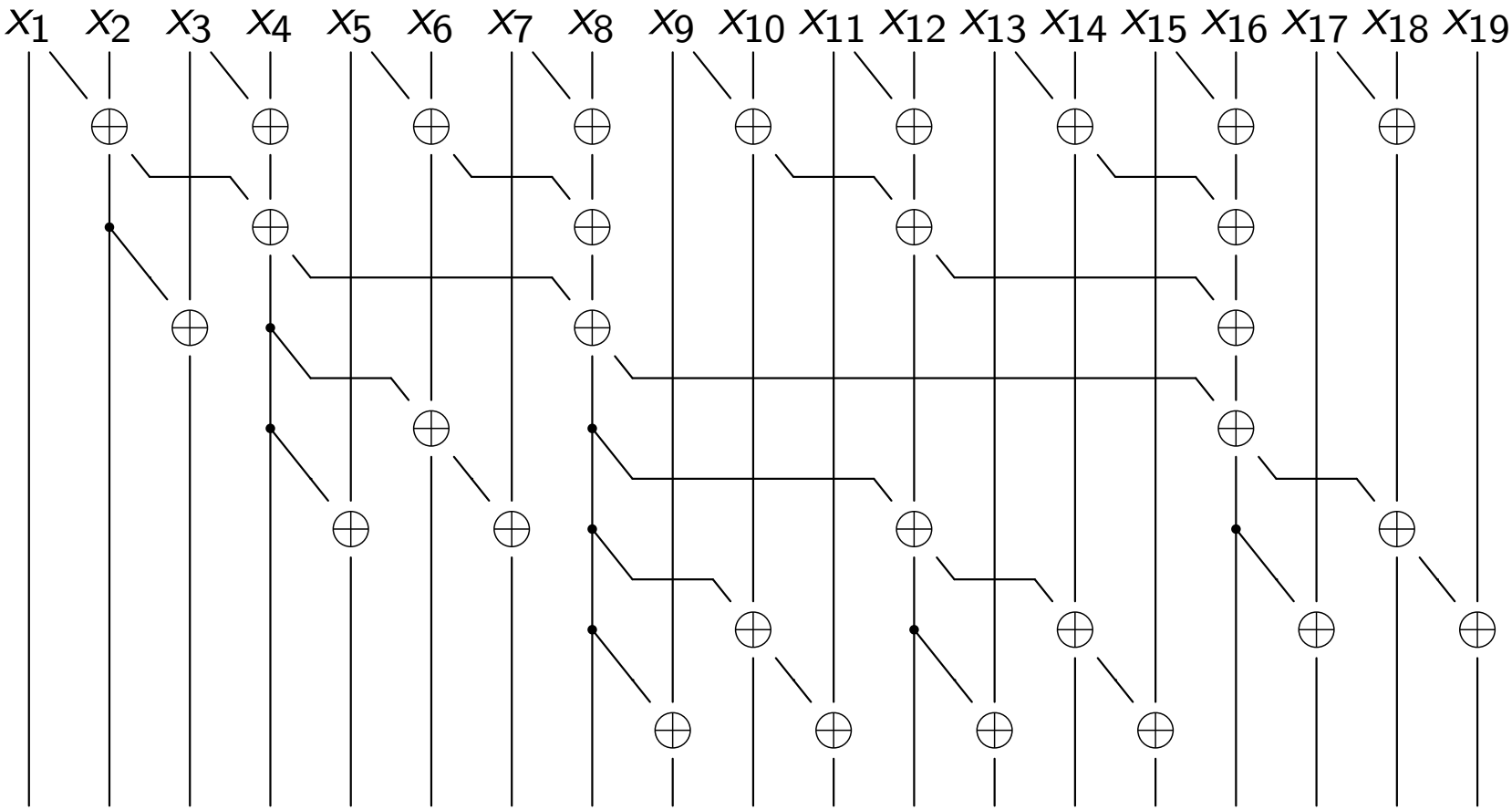

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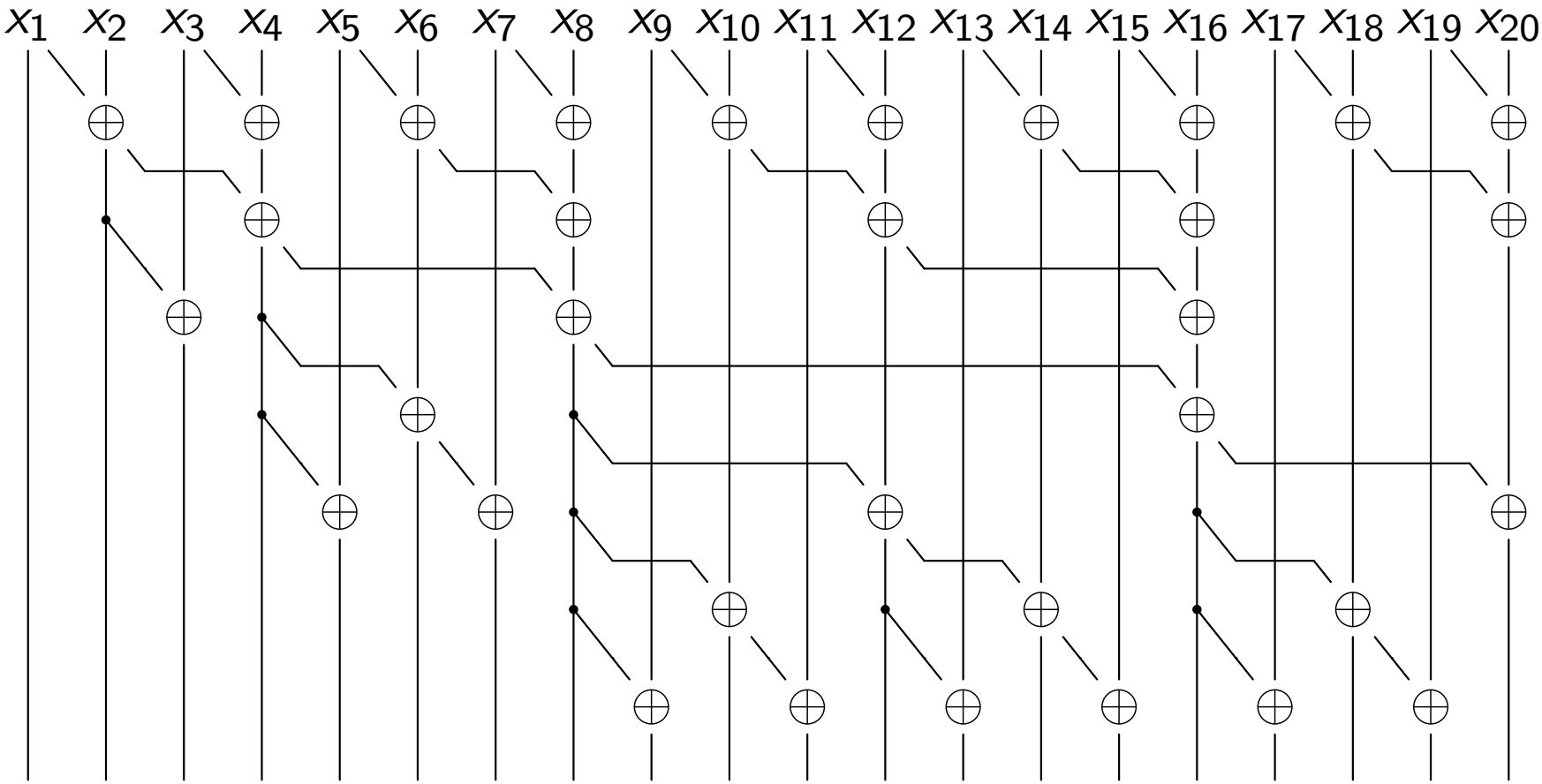

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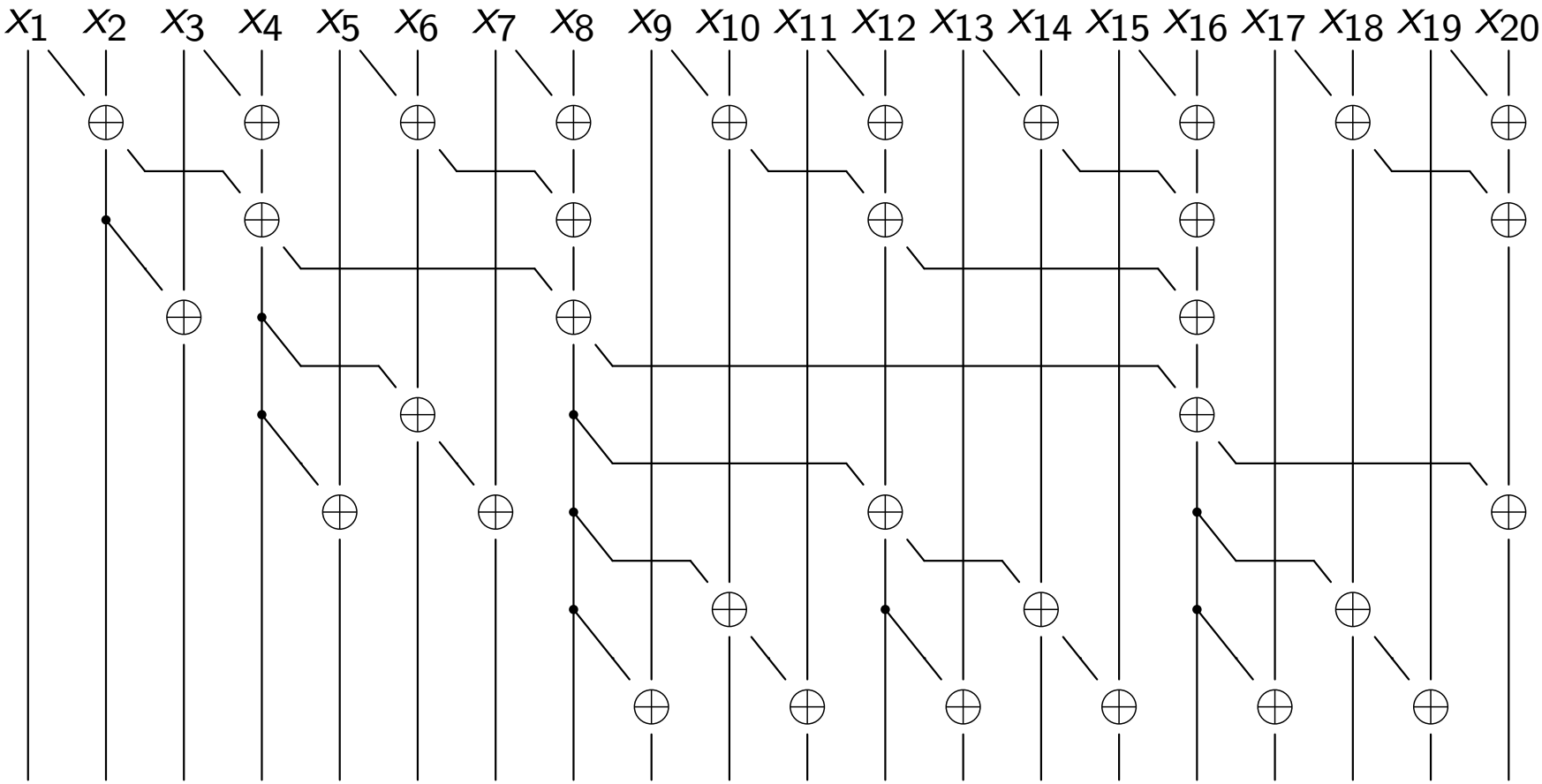

Wanted: reasoning principles, verification techniques, systematic testing approach, ...

Investigating Particular Instances Only

Knuth's 0-1-Principle

If a comparison-swap algorithm sorts correctly on the Booleans, it does so on arbitrary totally ordered value sets.

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Unfortunately not !

A Knuth-like 0-1-2-Principle [V. 2008]

Given: $\texttt{scanl1} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$

$$\texttt{scanl1}\ (\oplus)\ (x : xs) = go\ x\ xs$$
$$\textbf{where}\ go\ x\ [\,] = [x]$$
$$go\ x\ (y : ys) = x : (go\ (x \oplus y)\ ys)$$

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data Three = Zero | One | Two

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$$\textbf{data}\ \mathsf{Three} = \mathsf{Zero} \mid \mathsf{One} \mid \mathsf{Two}$$

Theorem: If for every $xs :: [\mathsf{Three}]$ and associative $(\oplus) :: \mathsf{Three} \to \mathsf{Three} \to \mathsf{Three}$,

$$\texttt{candidate}\ (\oplus)\ xs \quad = \quad \texttt{scanl1}\ (\oplus)\ xs\,,$$

then the same holds for every type τ, $xs :: [\tau]$, and associative $(\oplus) :: \tau \to \tau \to \tau$.

Why 0-1-2? And How?

A question: What can `candidate` :: $(\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$ do, given an operation $\oplus$ and input list $[x_1, \ldots, x_n]$?

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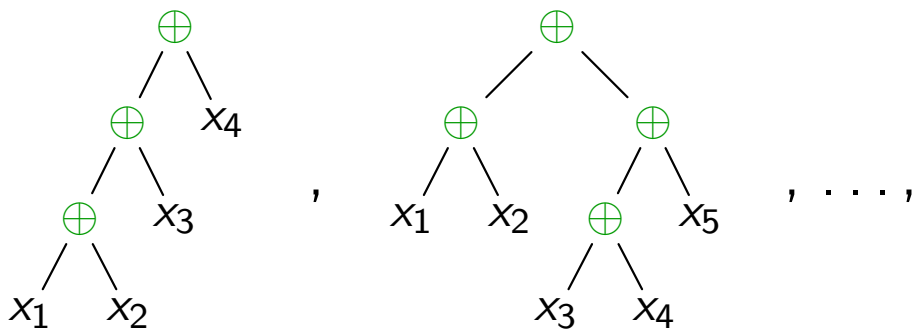
Among these expressions, there are good ones:



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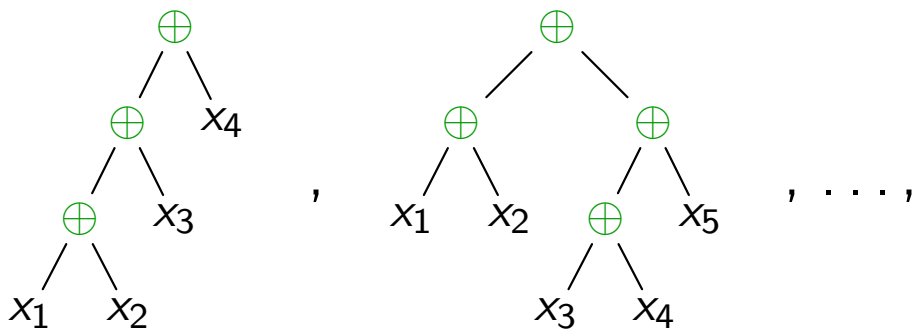
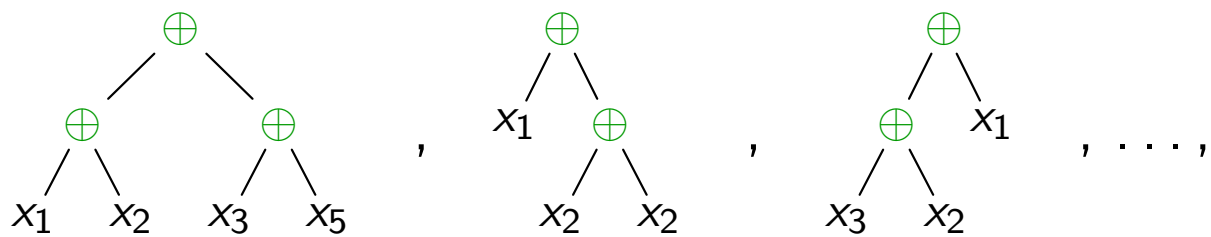
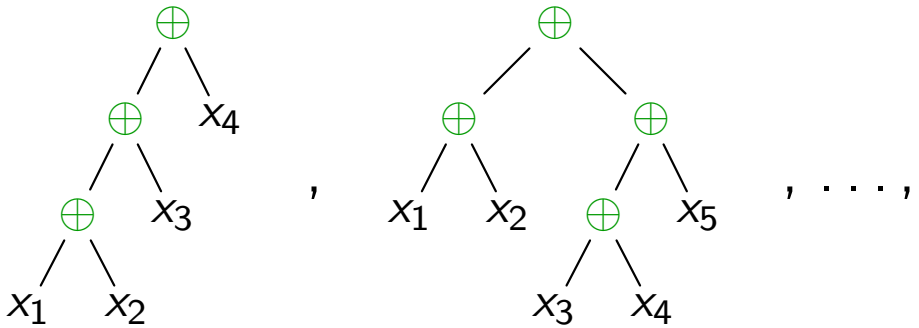


bad ones:

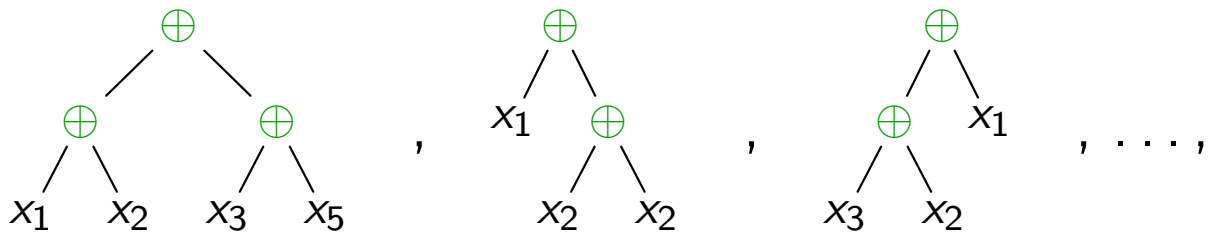


Why 0-1-2? And How?

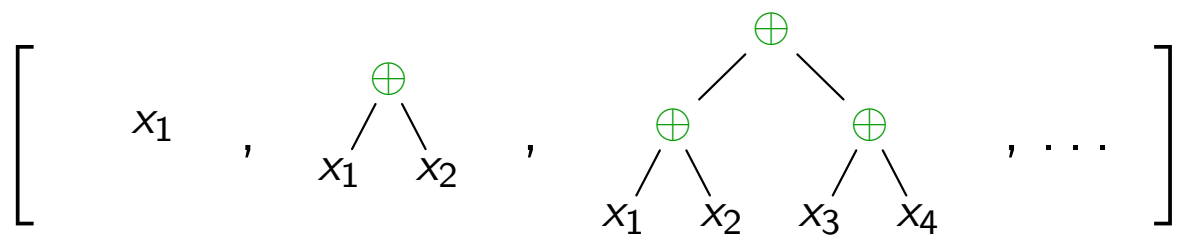
Among these expressions, there are good ones:



bad ones:



and ones in the wrong position:



That's How!

Let

$\oplus_1$	Zero	One	Two
Zero	Zero	One	Two
One	One	Two	Two
Two	Two	Two	Two

and

$\oplus_2$	Zero	One	Two
Zero	Zero	One	Two
One	One	One	Two
Two	Two	One	Two

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If `candidate` $(\oplus_1)$ is correct on each list of the form

$$[(\text{Zero},)^* \ \text{One} \ (, \text{Zero})^* \ (, \text{Two})^*]$$

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and `candidate` $(\oplus_2)$ is correct on each list of the form

$$[(\text{Zero},)^* \ \text{One}, \text{Two} \ (, \text{Zero})^*]$$

then `candidate` is correct for associative $\oplus$ at arbitrary type.

A Knuth-like 0-1-2-Principle [V. 2008]

Given: $\texttt{scanl1} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$

$$\begin{aligned}\texttt{scanl1}\ (\oplus)\ (x : xs) &= go\ x\ xs \\ \textbf{where}\ go\ x\ [\,] &= [x] \\ go\ x\ (y : ys) &= x : (go\ (x \oplus y)\ ys)\end{aligned}$$

$\texttt{candidate} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$

data Three = Zero | One | Two

Theorem: If for every xs :: [Three] and associative $(\oplus) ::$ Three $\to$ Three $\to$ Three,

$$\texttt{candidate}\ (\oplus)\ xs \quad = \quad \texttt{scanl1}\ (\oplus)\ xs\,,$$

then the same holds for every type τ, $xs :: [\tau]$, and associative $(\oplus) :: \tau \to \tau \to \tau$.

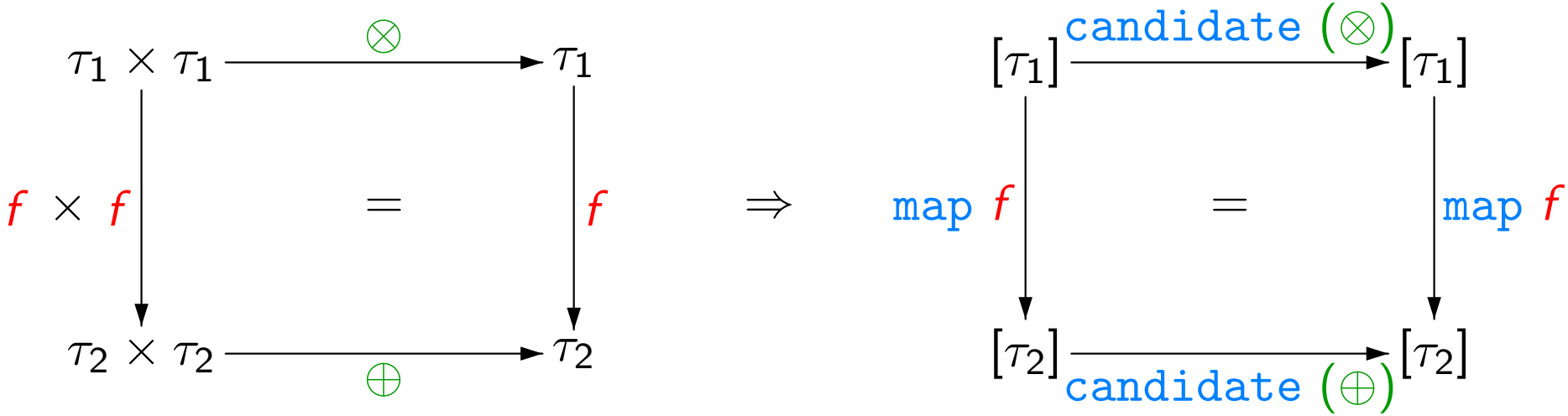
Using the Free Theorems Generator

Input:
```
candidate :: (a -> a -> a) -> [a] -> [a]
```

Output:
```
forall t1,t2 in TYPES, f :: t1 -> t2.
        forall g :: t1 -> t1 -> t1.
          forall h :: t2 -> t2 -> t2.
            (forall x :: t1. forall y :: t1.
               f (g x y) = h (f x) (f y))
            ==> (forall z :: [t1].
                   map f (candidate g z)
                    = candidate h (map f z))
```

Rephrased

For every choice of types τ_1, τ_2 and functions $f :: \tau_1 \to \tau_2$, $(\otimes) :: \tau_1 \to \tau_1 \to \tau_1$, and $(\oplus) :: \tau_2 \to \tau_2 \to \tau_2$:



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Given:
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$$go\ x\ (y : ys) = x : (go\ (x \oplus y)\ ys)$$

$$\texttt{candidate} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$$

$$\textbf{data}\ \mathsf{Three} = \mathsf{Zero} \mid \mathsf{One} \mid \mathsf{Two}$$

Theorem: If for every $xs :: [\mathsf{Three}]$ and associative $(\oplus) :: \mathsf{Three} \to \mathsf{Three} \to \mathsf{Three}$,

$$\texttt{candidate}\ (\oplus)\ xs \quad = \quad \texttt{scanl1}\ (\oplus)\ xs\,,$$

then the same holds for every type τ, $xs :: [\tau]$, and associative $(\oplus) :: \tau \to \tau \to \tau$.

Decomposing the 0-1-2-Principle

Proposition 1: If `candidate` $(\oplus_1)$ is correct on each list of the form $[(\mathsf{Zero},)^*\ \mathsf{One}\ (,\mathsf{Zero})^*\ (,\mathsf{Two})^*]$ and `candidate` $(\oplus_2)$ is correct on each list of the form $[(\mathsf{Zero},)^*\ \mathsf{One},\mathsf{Two}\ (,\mathsf{Zero})^*]$, then for every $n \geq 0$,
`candidate` $(+\!\!+)\ [[k] \mid k \leftarrow [0..n]] \ = \ [[0..k] \mid k \leftarrow [0..n]]$ (*).

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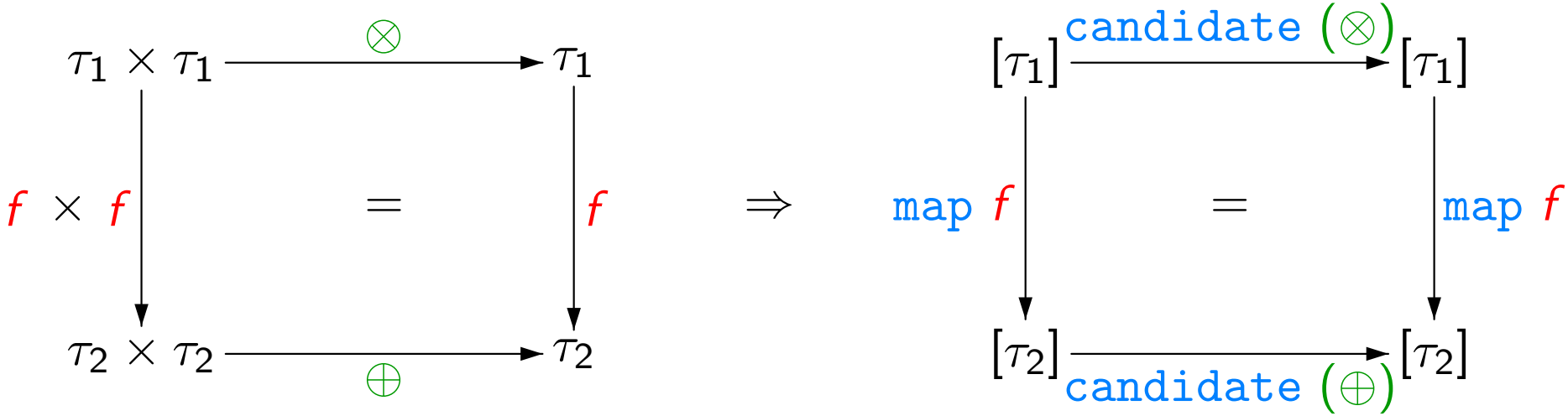

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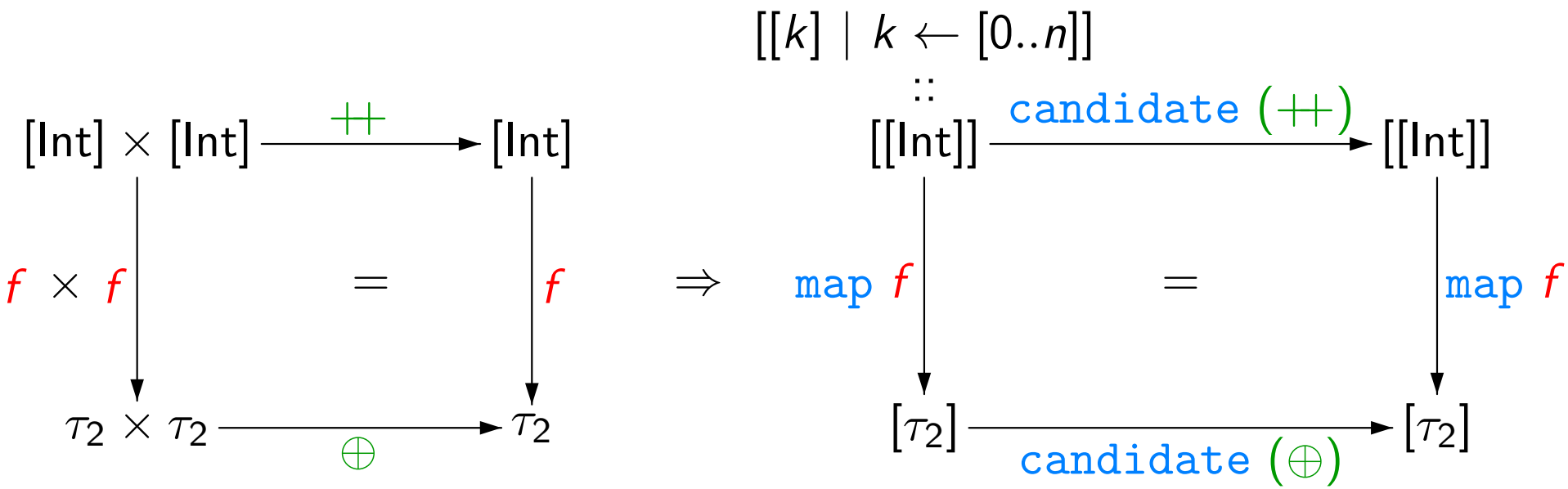

What Else?

- For parallel prefix computation, formalisation available in Isabelle/HOL [Böhme 2007].

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- There is still an interesting story to tell behind how "0-1-2", $\oplus_1$, $\oplus_2$, ... were found.
- For which other algorithm classes can one play the same trick?

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