New Applications of Parametricity

Janis Voigtländer

Technische Universität Dresden

ISS-AiPL'09

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News About

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Parametric Polymorphism in Haskell

A standard function:

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\begin{aligned}
\texttt{map}\ f\ [\,] &= [\,] \\
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$\texttt{map succ}$ $[1, 2, 3]$	$= [2, 3, 4]$		— $\alpha, \beta \mapsto \mathsf{Int}, \mathsf{Int}$
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Another Example

$$\begin{aligned}
&\texttt{reverse} :: [\alpha] \to [\alpha] \\
&\texttt{reverse}\ [\,] \hphantom{(a : as)} = [\,] \\
&\texttt{reverse}\ (a : as) = (\texttt{reverse}\ as) +\!\!+ [a]
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Another Example

$$\begin{aligned}
&\texttt{reverse} :: [\alpha] \rightarrow [\alpha] \\
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\end{aligned}$$

For every choice of f and l:

$$\texttt{reverse}\ (\texttt{map}\ f\ \ l) \quad = \quad \texttt{map}\ f\ (\texttt{reverse}\ l)$$

Provable by induction.

Another Example

$$\begin{array}{l} \texttt{reverse} :: [\alpha] \to [\alpha] \\ \texttt{reverse}\ [\,] \qquad\quad = [\,] \\ \texttt{reverse}\ (a : as) = (\texttt{reverse}\ as) + [a] \end{array}$$

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Or as a "free theorem" [Wadler, FPCA'89].

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Another Example

$$\texttt{reverse} :: [\alpha] \rightarrow [\alpha]$$

$$\texttt{tail} :: [\alpha] \rightarrow [\alpha]$$

For every choice of f and l:

$$\texttt{reverse}\ (\texttt{map}\ f\ \ l) \quad = \quad \texttt{map}\ f\ (\texttt{reverse}\ l)$$

$$\texttt{tail}\ (\texttt{map}\ f\ \ l) \quad = \quad \texttt{map}\ f\ (\texttt{tail}\ l)$$

Another Example

$$\begin{aligned} \texttt{reverse} &:: [\alpha] \to [\alpha] \\ \texttt{tail} &:: [\alpha] \to [\alpha] \\ \texttt{g} &:: [\alpha] \to [\alpha] \end{aligned}$$

For every choice of f and l:

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Automatic Generation of Free Theorems

At `http://linux.tcs.inf.tu-dresden.de/~voigt/ft`:

This tool allows to generate free theorems for sublanguages of Haskell as described here.

The source code of the underlying library and a shell-based application using it is available here and here.

Please enter a (polymorphic) type, e.g. "(a -> Bool) -> [a] -> [a]" or simply "filter":

```
g :: (a -> Bool) -> [a] -> [a]
```

Please choose a sublanguage of Haskell:

- (●) no bottoms (hence no general recursion and no selective strictness)
- ( ) general recursion but no selective strictness
- ( ) general recursion and selective strictness

Please choose a theorem style (without effect in the sublanguage with no bottoms):

- (●) equational
- ( ) inequational

Generate

Automatic Generation of Free Theorems

The theorem generated for functions of the type

```
g :: forall a . (a -> Bool) -> [a] -> [a]
```

in the sublanguage of Haskell with no bottoms is:

```
forall t1,t2 in TYPES, R in REL(t1,t2).
 forall p :: t1 -> Bool.
  forall q :: t2 -> Bool.
   (forall (x, y) in R. p x = q y)
   ==> (forall (z, v) in lift{[]}(R).
         (g p z, g q v) in lift{[]}(R))
```

The structural lifting occurring therein is defined as follows:

```
lift{[]}(R)
  = {([], [])}
  u {(x : xs, y : ys) |
       ((x, y) in R) && ((xs, ys) in lift{[]}(R))}
```

Reducing all permissible relation variables to functions yields:

```
forall t1,t2 in TYPES, f :: t1 -> t2.
 forall p :: t1 -> Bool.
  forall q :: t2 -> Bool.
   (forall x :: t1. p x = q (f x))
   ==> (forall y :: [t1]. map f (g p y) = g q (map f y))
```

Export as PDF   Show type instantiations   Enter a new type   Help page

Some Applications

- Short Cut Fusion [Gill et al., FPCA'93]

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- Bidirectionalisation [V., POPL'09]

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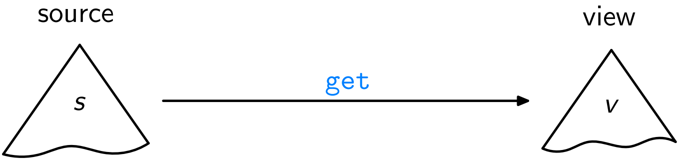
- Bidirectionalisation [V., POPL'09]
- Reasoning about invariants for monadic programs [V., ICFP'09]

Bidirectional Transformation



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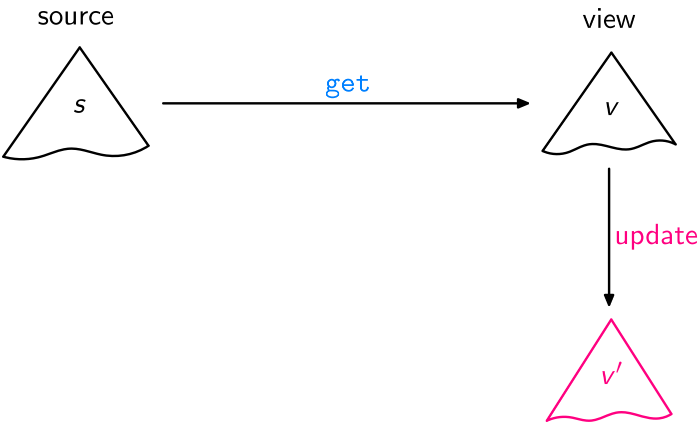

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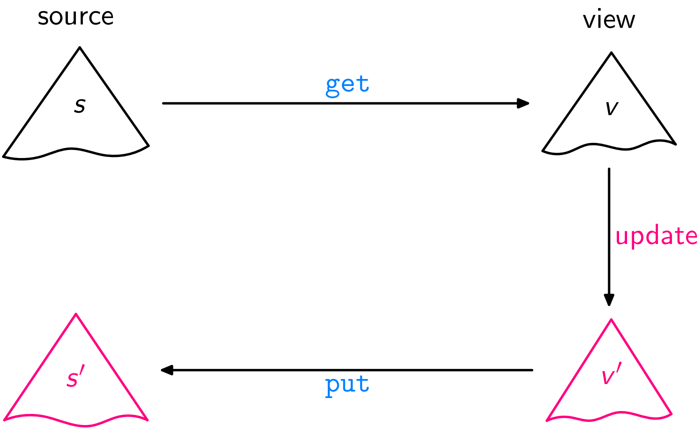

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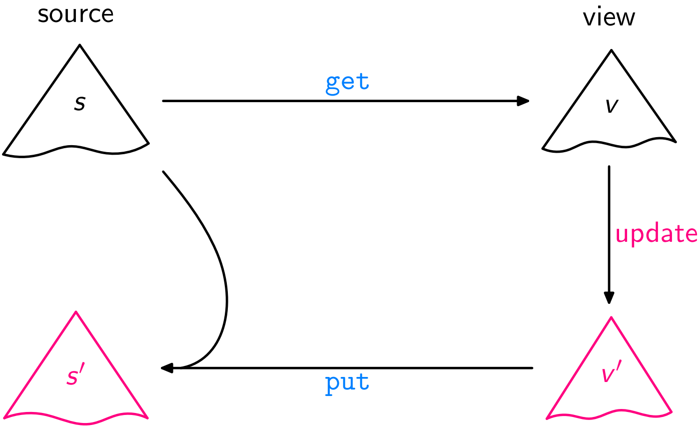

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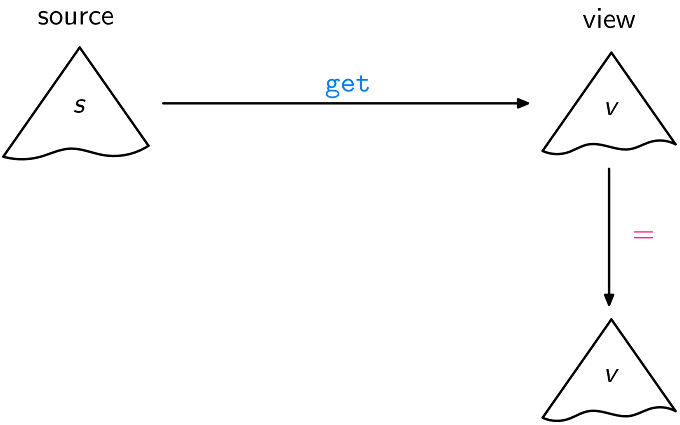
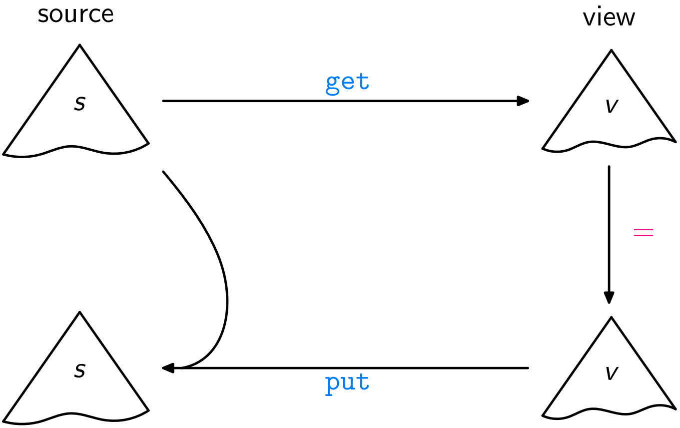


Acceptability / GetPut

Bidirectional Transformation



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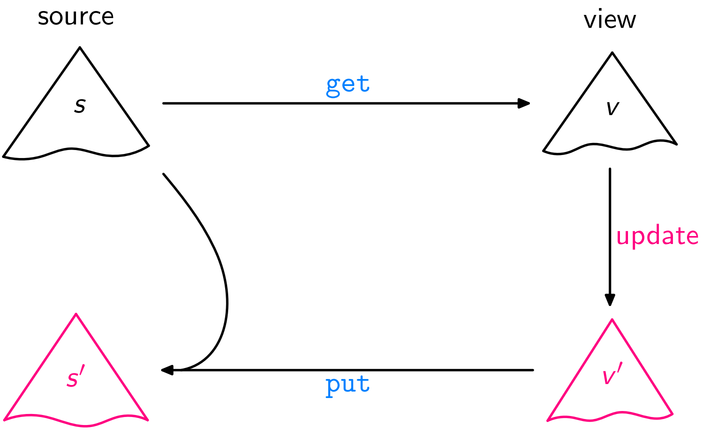

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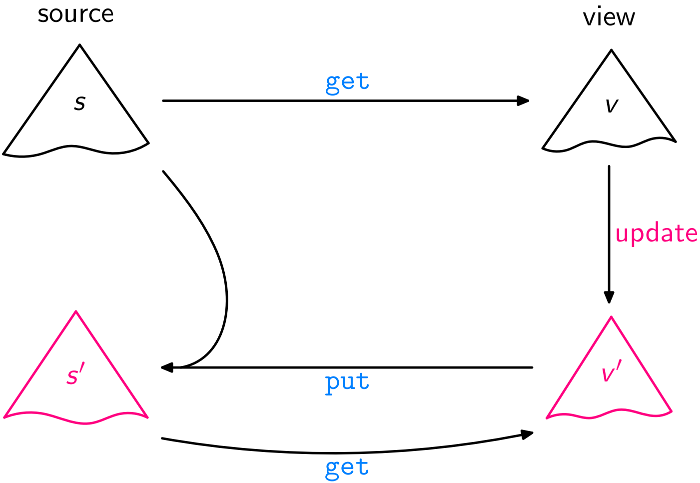

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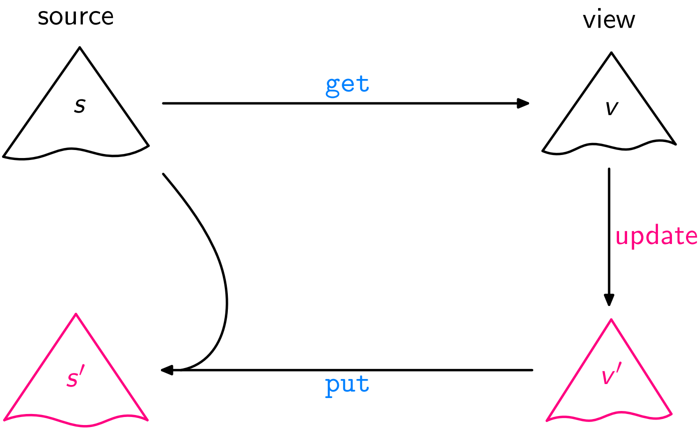

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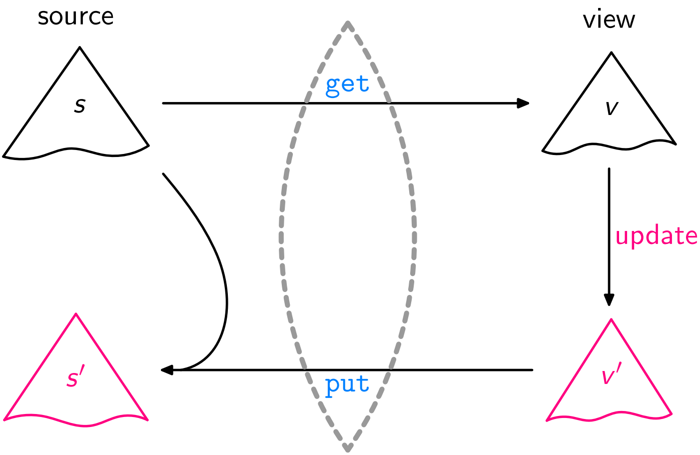
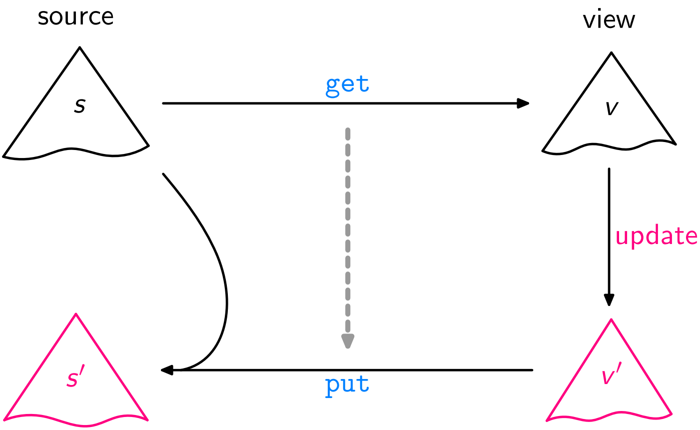


Lenses, DSLs

[Foster et al., ACM TOPLAS'07, ...]

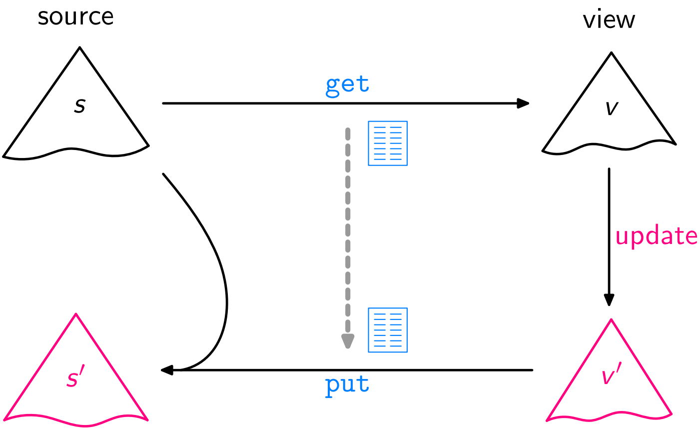
Bidirectional Transformation



Bidirectionalisation

[Matsuda et al., ICFP'07]

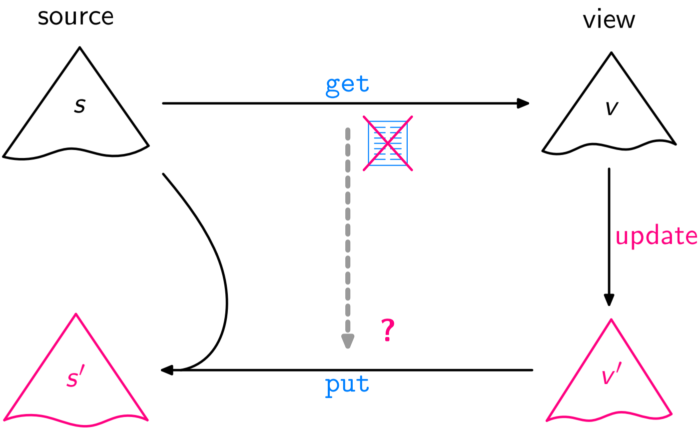
Bidirectional Transformation



Syntactic Bidirectionalisation

[Matsuda et al., ICFP'07]

Bidirectional Transformation



Semantic Bidirectionalisation

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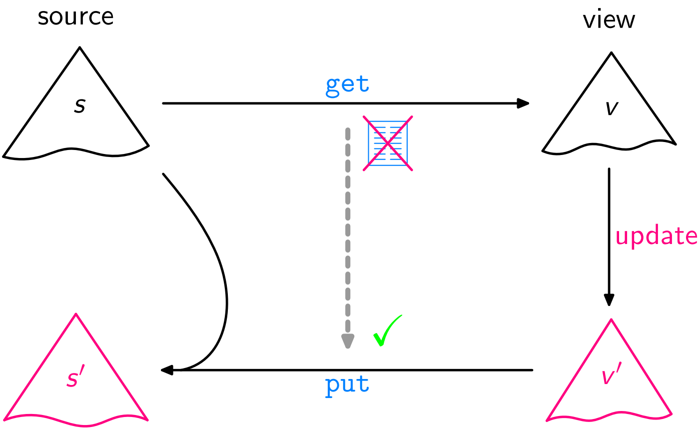

Semantic Bidirectionalisation

[V., POPL'09]

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Aim: Write a higher-order function `bff` such that any `get` and `bff get` satisfy GetPut, PutGet, ...

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Examples:

$$\text{“abc”} \xrightarrow{\texttt{tail}} \text{“bc”}$$

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Examples:

$$\begin{tikzcd}
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 & \text{“de”}
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$$
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\text{“abc”} \arrow[r, "\texttt{tail}"] \arrow[d, bend left] & \text{“bc”} \arrow[d, "\text{update}"] \\
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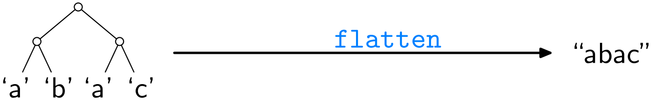

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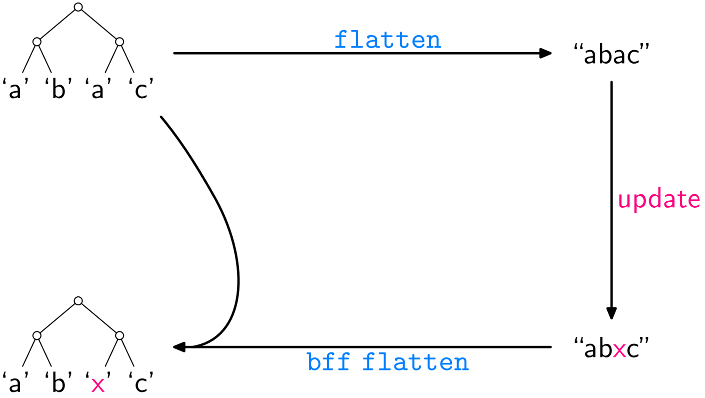

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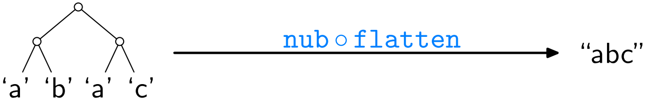

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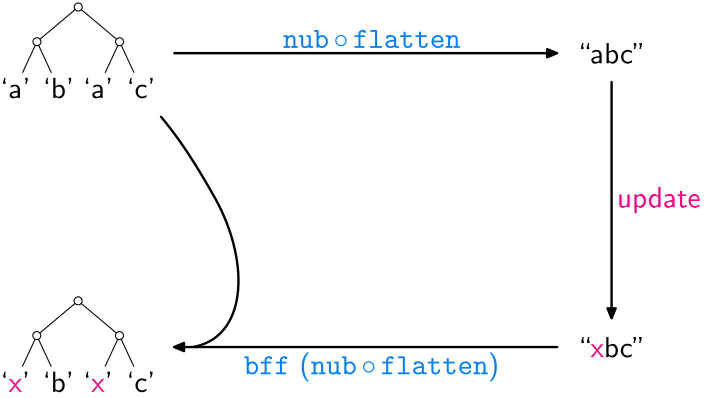

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Analysing Specific Instances

Assume we are given some

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How can we, or `bff`, analyse it without access to its source code?

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Like:

$$\texttt{get}\ [0..n] = \begin{cases} [1..n] & \text{if } \texttt{get} = \texttt{tail} \\ [n..0] & \text{if } \texttt{get} = \texttt{reverse} \\ [0..(\texttt{min}\ 4\ n)] & \text{if } \texttt{get} = \texttt{take}\ 5 \\ & \vdots \end{cases}$$

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Then transfer the gained insights to source lists other than $[0..n]$!

Using a Free Theorem

For every

$$\mathtt{g} :: [\alpha] \to [\alpha]$$

we have

$$\mathtt{map}\ f\ (\mathtt{g}\ l) \quad = \quad \mathtt{g}\ (\mathtt{map}\ f\ l)$$

for arbitrary f and l.

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Given an arbitrary list s of length $n + 1$, set $\mathtt{g} = \mathtt{get}$, $l = [0..n]$, $f = (s\,!!)$, leading to:

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$$\texttt{inv} :: (V, V^C) \rightarrow S$$

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$$\texttt{inv} :: (V, V^C) \to S$$

Then:

$$\begin{array}{l} \texttt{put} :: S \to V \to S \\ \texttt{put}\ s\ v' = \texttt{inv}\ (v', \texttt{compl}\ s) \end{array}$$

The Constant-Complement Approach [Bancilhon & Spyratos, ACM TODS'81]

In general, given

$$\texttt{get} :: S \to V$$

define a V^C and

$$\texttt{compl} :: S \to V^C$$

such that

$$\lambda s \to (\texttt{get}\ s, \texttt{compl}\ s)$$

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Then:

$$\begin{aligned} &\texttt{put} :: S \to V \to S \\ &\texttt{put}\ s\ v' = \texttt{inv}\ (v', \texttt{compl}\ s) \end{aligned}$$

Important: `compl` should "collapse" as much as possible.

The Constant-Complement Approach

For our setting,

$$\texttt{get} :: [\alpha] \to [\alpha]\,,$$

what should be V^C and

$$\texttt{compl} :: [\alpha] \to V^C \quad ???$$

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For the moment, be maximally conservative.

The Complement Function

$$\textbf{type}\ \mathsf{IntMap}\ \alpha = [(\mathsf{Int}, \alpha)]$$

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\begin{array}{l}
\texttt{compl} :: [\alpha] \to (\mathsf{Int}, \mathsf{IntMap}\ \alpha) \\
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For example:

$$\texttt{get} = \texttt{tail} \qquad \rightsquigarrow \qquad \texttt{compl}\ \text{“abcde”} = (5, [(0, \text{'a'})])$$

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$$\texttt{get} = \texttt{reverse} \quad \rightsquigarrow \quad \texttt{compl}\ \text{“abcde”} = (5, [\,])$$

An Inverse of $\lambda s \rightarrow (\texttt{get}\ s, \texttt{compl}\ s)$

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\begin{array}{l}
\texttt{inv} :: ([\alpha], (\mathsf{Int}, \mathsf{IntMap}\ \alpha)) \rightarrow [\alpha] \\
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† Can be thought of as `zip` for the moment.

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To prove formally:

- $\mathtt{inv}\ (\mathtt{get}\ s, \mathtt{compl}\ s) = s$
- if $\mathtt{inv}\ (v, c)$ defined, then $\mathtt{get}\ (\mathtt{inv}\ (v, c)) = v$
- if $\mathtt{inv}\ (v, c)$ defined, then $\mathtt{compl}\ (\mathtt{inv}\ (v, c)) = c$

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Altogether:

$$\textbf{type}\ \mathsf{IntMap}\ \alpha = [(\mathsf{Int}, \alpha)]$$

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$$\begin{array}{l}
\texttt{put} :: [\alpha] \to [\alpha] \to [\alpha] \\
\texttt{put}\ s\ v' = \texttt{inv}\ (v', \texttt{compl}\ s)
\end{array}$$

“Fusion”

Inlining `compl` and `inv` into `put`:

$$
\begin{aligned}
\texttt{put}\ s\ v' = \textbf{let}\ & n\ = (\texttt{length}\ s) - 1 \\
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&\texttt{assoc}\ [\,]\ \ \ \ \ \ \ \ [\,] = [\,] \\
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&\qquad\qquad \text{Nothing} \rightarrow (i, b) : m \\
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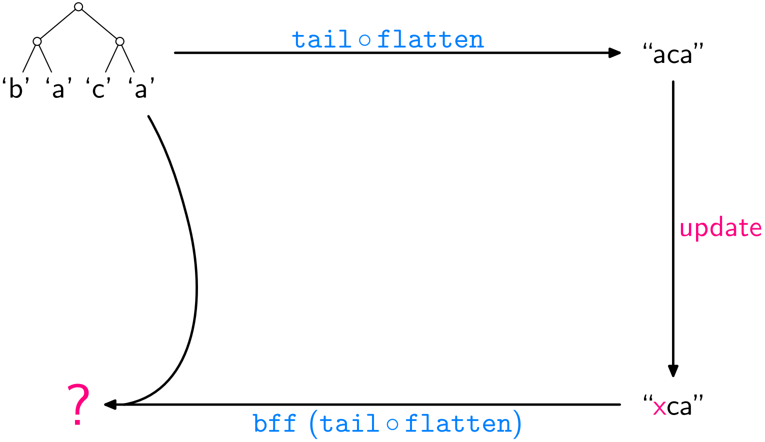
$$

Actual code only slightly more elaborate!

The Resulting Bidirectionalisation Method in Action



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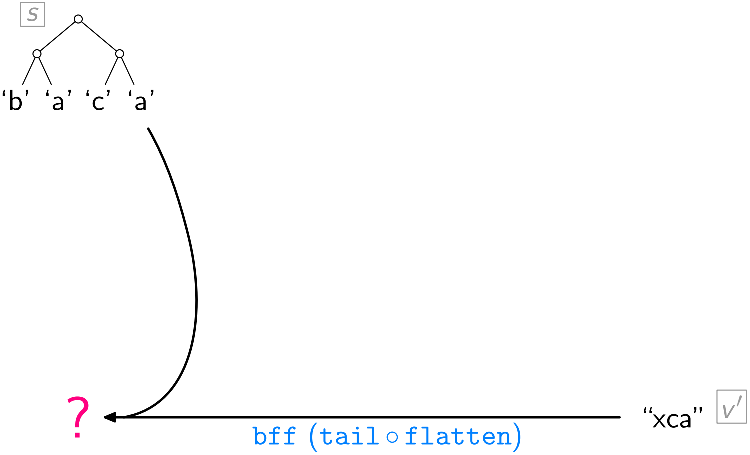

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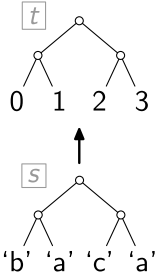

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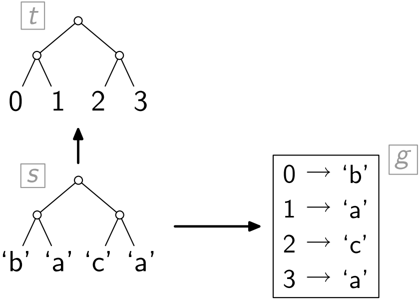

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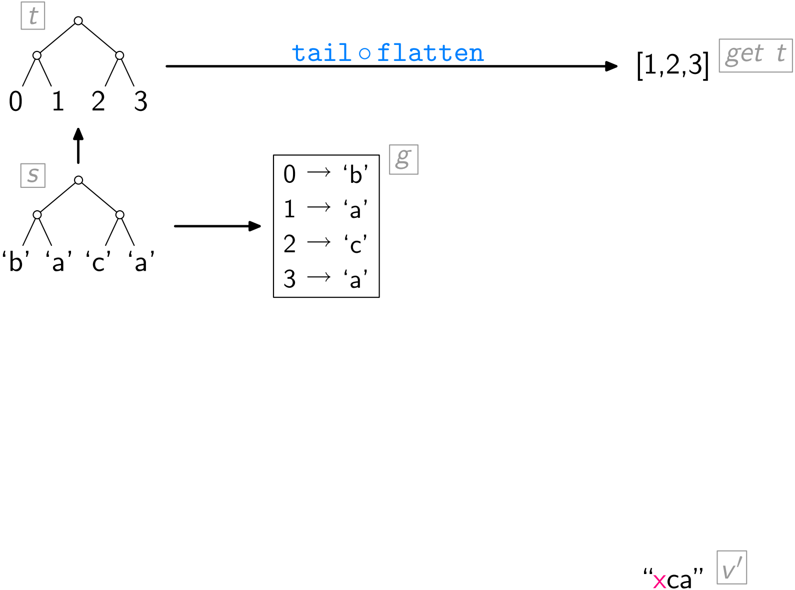

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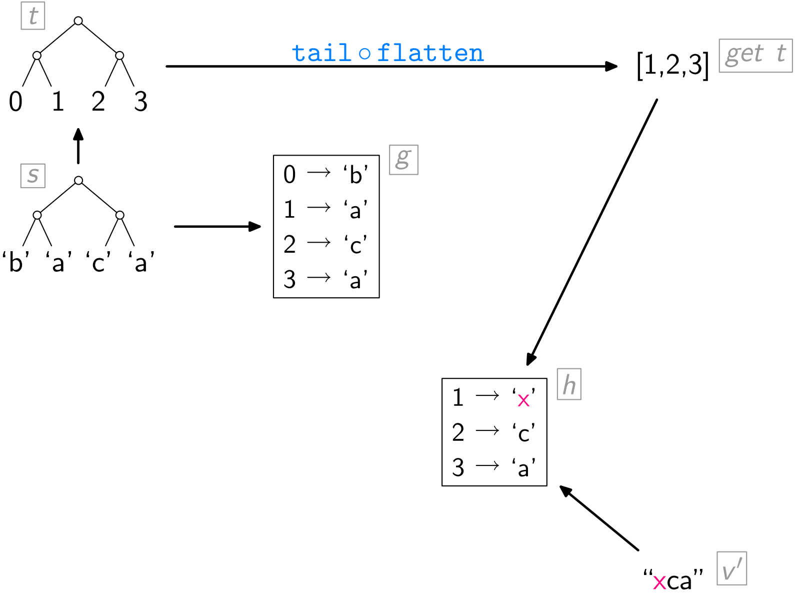

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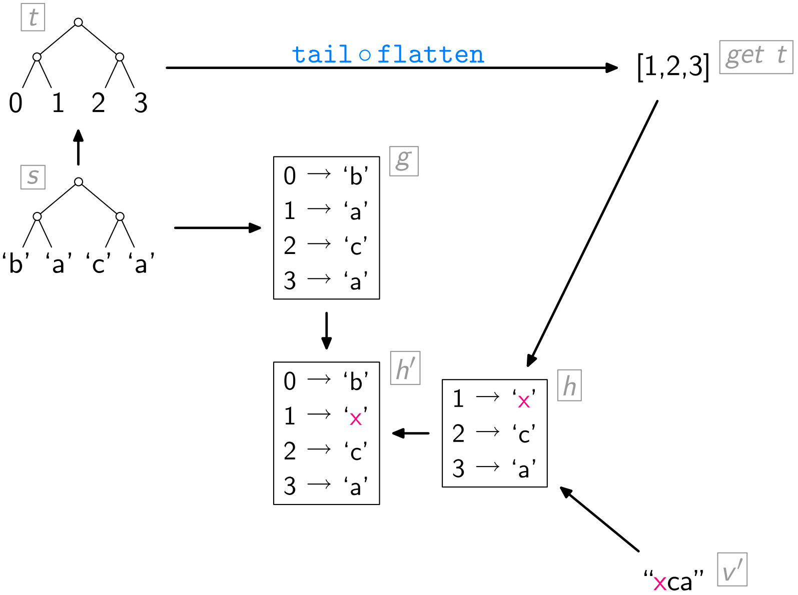

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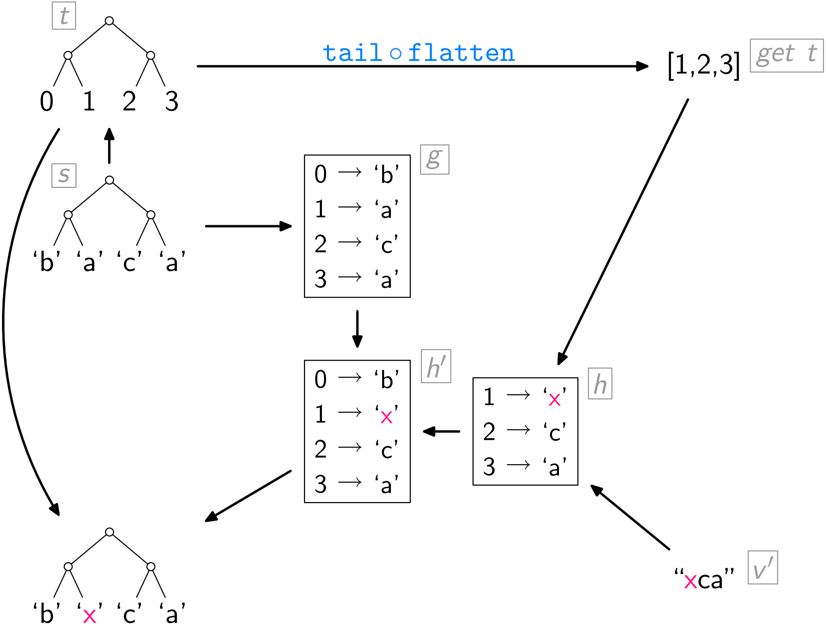

Extending the Technique

Major Problem:

- Shape-affecting updates lead to failure.

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- Being maximally conservative this way often does not "collapse enough".

- For example:

  $\texttt{get} = \texttt{tail} \quad \rightsquigarrow \quad$ `put` "abcde" "xyz" fails precisely because
  `compl` "abcde" $= (5, [(0, \text{'a'})])$

Assuming Shape-Injectivity

So assume there is a function

$$\texttt{shapeInv} :: \mathsf{Int} \to \mathsf{Int}$$

with, for every source list s,

$$\texttt{length}\ s = \texttt{shapeInv}\ (\texttt{length}\ (\texttt{get}\ s))$$

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But how to obtain `shapeInv` ???

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But how to obtain `shapeInv` ???

One possibility: provided by user.

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Just for experimentation:

$\texttt{shapeInv} :: \mathrm{Int} \to \mathrm{Int}$
$\texttt{shapeInv}\ l = \texttt{head}\ [n+1 \mid n \leftarrow [0..],\ (\texttt{length}\ (\texttt{get}\ [0..n])) == l]$

Not Quite There, Yet

Works quite nicely in some cases:

$\texttt{get} = \texttt{tail} \quad \rightsquigarrow \quad \texttt{put}$ “abcde” “xyz” $=$ “axyz”, using
$\texttt{compl}$ “abcde” $= [(0, \text{'a'})]$

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The problem: by keeping indices around, `compl` still does not "collapse enough".

Note: even without these indices, $\lambda s \rightarrow (\texttt{get}\ s, \texttt{compl}\ s)$ would be injective.

Eliminating Indices

$$
\begin{aligned}
&\texttt{compl} :: [\alpha] \to [(\mathsf{Int}, \alpha)] \\
&\texttt{compl}\ s = \mathbf{let}\ \begin{aligned}[t] n &= (\texttt{length}\ s) - 1 \\ t &= [0..n] \\ g &= \texttt{zip}\ t\ s \\ g' &= \texttt{filter}\ (\lambda(i, \_) \to \texttt{notElem}\ i\ (\texttt{get}\ t))\ g \end{aligned} \\
&\qquad\qquad\ \mathbf{in}\ \ g'
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\qquad\qquad\qquad\qquad h' = h + g' \\
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Now:

$$\texttt{get} = \texttt{init} \quad \leadsto \quad \texttt{put}\ \text{"abcde"}\ \ \text{"xyz"} = \text{"xyze"}$$

More Examples

Let $\texttt{get} = \texttt{sieve}$ with:

$$
\begin{aligned}
&\texttt{sieve} :: [\alpha] \rightarrow [\alpha] \\
&\texttt{sieve}\ (a : b : cs) = b : (\texttt{sieve}\ cs) \\
&\texttt{sieve}\ \_ \qquad\qquad\quad = []
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Then:

$$\texttt{put}\ [1..8]\ [2, -4, 6, 8] \quad = \quad [1, 2, 3, -4, 5, 6, 7, 8]$$

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However:

$$\texttt{put}\ [1..8]\ [0, 2, -4, 6, 8] \quad = \quad [1, 0, 3, 2, 5, -4, 7, 6, \perp, 8]$$

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However:

$$\texttt{put}\ [1..8]\ [0, 2, -4, 6, 8] = [1, 0, 3, 2, 5, -4, 7, 6, \bot, 8]$$

Whereas we might have preferred:

$$\texttt{put}\ [1..8]\ [0, 2, -4, 6, 8] = [\bot, 0, 1, 2, 3, -4, 5, 6, 7, 8]$$

Conclusion

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- constrain the behaviour of programs

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- push towards full programming languages
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